Continuous Symmetries

February 23, 2017

1 Unitary transformations as symmetries of quantum mechanics

Consider an arbitrary linear transformation of a state $|\alpha\rangle$,

$$|\tilde{\alpha}\rangle = \hat{\mathcal{U}}\,|\alpha\rangle$$

If $\hat{\mathcal{O}}$ is to be a symmetry of a quantum system, it must preserve physical probabilities, and therefore, it must preserve the norms of all states, Therefore, for symmetries we demand

$$\begin{aligned} \langle\tilde{\alpha} \mid \tilde{\alpha}\rangle &= \langle\alpha \mid \alpha\rangle \\ \langle\alpha|\hat{\mathcal{U}}^{\dagger}\hat{\mathcal{U}}\,|\alpha\rangle &= \langle\alpha \mid \alpha\rangle \end{aligned}$$

This, in turn, requires

$$\hat{\mathcal{U}}^{\dagger}\hat{\mathcal{U}} = 1$$

so $\hat{\mathcal{U}}$ preserves norms if and only if it is unitary.

The action of a general transformations on the Schrödinger equation,

$$\hat{H}\,|\psi\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi\rangle$$

is found by acting with any time-independent operator $\hat{\mathcal{O}}$ on both sides,

$$\hat{\mathcal{O}}\hat{H}\,|\psi\rangle = i\hbar\frac{\partial}{\partial t}\hat{\mathcal{O}}\,|\psi\rangle$$

Then we may insert the identity operator, $1 = \hat{\mathcal{O}}^{-1}\hat{\mathcal{O}}$ between the Hamiltonian and the state

$$\hat{\mathcal{O}}\hat{H}\hat{\mathcal{O}}^{-1}\hat{\mathcal{O}}\,|\psi\rangle = i\hbar\frac{\partial}{\partial t}\hat{\mathcal{O}}\,|\psi\rangle$$

The fprm Schrödinger equation, and indeed any vector equation, is preserved by on the new state $\left|\tilde{\psi}\right\rangle \equiv \hat{\mathcal{O}}\,|\psi\rangle$ as long as all linear operators are simultaneously transformed by a *similarity transformation*,

$$\tilde{\hat{H}} \equiv \hat{\mathcal{O}}\hat{H}\hat{\mathcal{O}}^{-1}$$

By a *symmetry of a quantum system* we mean a transformation that takes solutions of the Schrödinger equation to other solutions of the same equation. Such a transformation must leave the Hamiltonian *invariant*,

$$\hat{H} = \hat{\mathcal{O}}\hat{H}\hat{\mathcal{O}}^{-1}$$

For quantum mechanics, we require more, since we must preserve Hermiticity of all observables. We therefore demand

$$\tilde{\hat{A}}^{\dagger} \equiv \tilde{\hat{A}}$$

whenever

$$\hat{A}^{\dagger} \equiv \hat{A}$$

As a consequence,

$$\begin{aligned}
\left(\hat{\mathcal{O}}\hat{A}\hat{\mathcal{O}}^{-1}\right)^{\dagger} &= \hat{\mathcal{O}}\hat{A}\hat{\mathcal{O}}^{-1} \\
\left(\hat{\mathcal{O}}^{-1}\right)^{\dagger}\hat{A}^{\dagger}\hat{\mathcal{O}}^{\dagger} &= \hat{\mathcal{O}}\hat{A}\hat{\mathcal{O}}^{-1} \\
\left(\hat{\mathcal{O}}^{-1}\right)^{\dagger}\hat{A}\hat{\mathcal{O}}^{\dagger} &= \hat{\mathcal{O}}\hat{A}\hat{\mathcal{O}}^{-1}
\end{aligned}$$

Multiplying on the right by $\hat{\mathcal{O}}$ and on the left by $\hat{\mathcal{O}}^{-1}$,

$$\hat{A} = \left(\hat{\mathcal{O}}^{-1}\left(\hat{\mathcal{O}}^{-1}\right)^{\dagger}\right)\hat{A}\left(\hat{\mathcal{O}}^{\dagger}\hat{\mathcal{O}}\right)$$

and since only the identity transformation leaves every operator invariant, we must have $\hat{\mathcal{O}}^{\dagger}\hat{\mathcal{O}} = 1$, and again find that $\hat{\mathcal{O}}$ must be unitary.

A symmetry of a quantum system is therefore given by any unitary transformation of the states that leaves the Hamiltonian invariant:

$$\begin{aligned}
\left|\tilde{\alpha}\right\rangle &= \hat{\mathcal{U}}\left|\alpha\right\rangle \\
\hat{H} &= \hat{\mathcal{U}}\hat{H}\hat{\mathcal{U}}^{\dagger}
\end{aligned}$$

2 Properties of continuous symmetries

A *continuous symmetry* is a family of unitary symmetries which may be parameterized by a finite number $n \geq 1$ of continuous parameters

$$\hat{\mathcal{U}} = \hat{\mathcal{U}}\left(\lambda_1, \lambda_2, \ldots, \lambda_n\right) = \hat{\mathcal{U}}\left(\boldsymbol{\lambda}\right)$$

We identify a few properties possessed or required by any such family, $\hat{\mathcal{U}}\left(\boldsymbol{\lambda}\right)$:

1. (Closure) Since we may follow one transformation by another, the product of any two transformations should be equivalent to another member of the family,

   $$\hat{\mathcal{U}}\left(\boldsymbol{\lambda}\right)\hat{\mathcal{U}}\left(\boldsymbol{\lambda}'\right) = \hat{\mathcal{U}}\left(\boldsymbol{\lambda}''\right)$$

   for some $\boldsymbol{\lambda}, \boldsymbol{\lambda}', \boldsymbol{\lambda}''$.

2. (Inverse) Since $\hat{\mathcal{U}}\left(\lambda\right)$ is unitary, its inverse exists and is given by $\hat{\mathcal{U}}^{\dagger}\left(\boldsymbol{\lambda}\right)$.

3. (Identity) From properties 1 and 2, we see that the identity is one of our transformations. It is convenitent to choose the parameters in such a way that $\hat{\mathcal{U}}\left(\mathbf{0}\right)$ is the identity transformation.

4. (Associativity) Matrix multiplication is associative; we require this property in general.

These properties are a natural consequence of asking for a symmetry. Any set of objects with the properties 1-4 is called a *group*, and with the continuous parameterization is called a *Lie group* (pronounced "lee").

For example, $\hat{\mathcal{U}}$ might the set of rotations of a plane

$$\hat{\mathcal{U}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We easily chech the group properties. Setting $\theta = 0$ gives the identity, while the inverse rotation is given by replacing θ by $-\theta$. Matrix multiplication is always associative, and we know that two sequential rotations by θ_1 and θ_2 are equivalent to a single rotation through $\theta_1 + \theta_2$. Therefore, rotations of the plane form a Lie group.

Three-dimensional rotations also form a group, and we will study it in great detail. In this case there are three parameters.

3 Continuous symmetries and observables

Corresponding to any continuous symmetry, there is a corresponding conserved observable. Consider a continuous family of unitary transformations, $\hat{\mathcal{U}}(\boldsymbol{\lambda})$. Let one parameter become infinitesmal while the others, if any, vanish: $\boldsymbol{\lambda} = (0, 0, \ldots, \varepsilon, 0, \ldots, 0)$. Then we may write $\hat{\mathcal{U}}$ to first order as

$$\hat{\mathcal{U}} = 1 - \frac{i}{\hbar}\varepsilon\hat{G}$$

with the infinitesimal parameter written as $-\frac{i}{\hbar}\varepsilon$ for later convenience. To first order the inverse is $\hat{\mathcal{U}}^{-1} = 1 + \frac{i\varepsilon}{\hbar}\hat{G}$ since then

$$\begin{aligned}
\hat{\mathcal{U}}\hat{\mathcal{U}}^{-1} &= \left(1 - \frac{i}{\hbar}\varepsilon\hat{G}\right)\left(1 + \frac{i\varepsilon}{\hbar}\hat{G}\right) \\
&= 1 - \frac{i}{\hbar}\varepsilon\hat{G} + \frac{i\varepsilon}{\hbar}\hat{G} + \frac{\varepsilon^2}{\hbar^2}\hat{G}\hat{G} \\
&= 1 + O\left(\varepsilon^2\right)
\end{aligned}$$

while the adjoint is

$$\hat{\mathcal{U}}^\dagger = 1 + \frac{i\varepsilon}{\hbar}\hat{G}^\dagger$$

Unitarity $\hat{\mathcal{U}}^\dagger = \hat{\mathcal{U}}^{-1}$, implies

$$\begin{aligned}
1 + \frac{i\varepsilon}{\hbar}\hat{G}^\dagger &= 1 + \frac{i\varepsilon}{\hbar}\hat{G} \\
\hat{G}^\dagger &= \hat{G}
\end{aligned}$$

so G is Hermitian. G is called an *infinitesimal generator* of the group of transformations.

We may repeat this procedure for each of the parameters of the family. This will give one infinitesimal generator for each parameter of the transformation group. Writing the parameters as λ^i for $i = 1, 2, \ldots, n$, we write the full set of generators as

$$G_i \quad i = 1, 2, \ldots, n$$

The set of all real linear combinations of the generators, $V = \left\{\sum_{i=1}^n a^i G_i \mid a^i \in \mathbb{R}^n\right\}$ is a vector space called the *Lie algebra* of the transformation group.

If $\hat{\mathcal{U}}$ is a symmetry of the Hamiltonian we then have for each generator,

$$\begin{aligned}
\left(1 - \frac{i\varepsilon}{\hbar}\hat{G}_i\right)\hat{H}\left(1 + \frac{i\varepsilon}{\hbar}\hat{G}_i\right) &= \hat{H} \\
\left[\hat{H}, \hat{G}_i\right] &= 0
\end{aligned}$$

This holds for the entire Lie algebra, since if $\hat{A} = \sum_{i=1}^n a^i\hat{G}_i = a^i\hat{G}_i$ for any real vector $\mathbf{a}$,

$$\begin{aligned}
\left[\hat{H}, \hat{A}\right] &= \left[\hat{H}, a^i\hat{G}_i\right] \\
&= a^i\left[\hat{H}, \hat{G}_i\right] \\
&= 0
\end{aligned}$$

If $\hat{\mathcal{U}}$ has no explicit time dependence, $\frac{\partial\hat{\mathcal{U}}}{\partial t} = 0$, then we will show in a subsequent Note that

$$\frac{d\hat{G}}{dt} = \left[\hat{H}, \hat{G}\right] = 0$$

so that the observable modeled by $\hat{G}$ is conserved.

Since $\hat{G}$ and $\hat{H}$ commute, we can form simultaneous eigenkets,

$$\begin{aligned}\hat{H}\left|E,g\right\rangle &= E\left|E,g\right\rangle \\ \hat{G}\left|E,g\right\rangle &= g\left|E,g\right\rangle\end{aligned}$$

and since the time evolution operator is built from the Hamiltonian, the simultaneous eigenkets remain simulataneous eigenkets.

Suppose that for some energy eigenket, $\left|E\right\rangle$, the transformation gives a distinct state,

$$\hat{\mathcal{U}}\left|E\right\rangle \neq \left|E\right\rangle$$

Then since $\hat{\mathcal{U}}$ commutes with $\hat{H}$, these distinct states have the same energy, so the energy is degenerate.

We will examine the cases of translations and rotations in detail, as well as the discrete symmetries of parity and time reversal. We begin here by studying translations.

4 Translations

Given any function of position, $f(\mathbf{x})$, we can consider a transformation to new coordinates, $\mathbf{y} = \boldsymbol{x} + \boldsymbol{a}$ for any constant vector $\mathbf{a}$. Such a coordinate transformation is called a *translation.* Classically, spatial translations preserve the Hamiltonian of a free particle $H(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}$ and therefore constitute a classical symmetry. Other classical Hamiltonians are invariant under certain translations, for example, a spring potential along the x-axis, $V = \frac{1}{2}kx^2$ is invariant under translations in the y or z directions. We will see that similar results hold in the quantum realm.

We now find the form of the translation operator and its associated conserved observable.

4.1 Basic properties of the translation operator

The basic properties of the translation operator, $\hat{\mathcal{T}}(\mathbf{a})$, are:

1. Translations commute:
$$\left[\hat{\mathcal{T}}(\mathbf{a}), \hat{\mathcal{T}}(\mathbf{b})\right] = 0$$

2. The inverse of any given translation is a translation by the opposite vector,
$$\hat{\mathcal{T}}^{-1}(\mathbf{a}) = \hat{\mathcal{T}}(-\mathbf{a})$$

3. To be a symmetry, the translation operator must be unitary,
$$\hat{\mathcal{T}}^{\dagger}(\mathbf{a})\,\hat{\mathcal{T}}(\mathbf{a}) = \hat{1}$$

To begin, write $\hat{\mathcal{T}}(\mathbf{a})$ as an exponential,

$$\hat{\mathcal{T}}(\mathbf{a}) = e^{-i\hat{\mathcal{O}}(\mathbf{a})}$$

In this form, $\hat{\mathcal{T}}(\mathbf{a})$ is unitary whenever $\hat{\mathcal{O}}$ is Hermitian, since then

$$\begin{aligned}\hat{\mathcal{T}}^{\dagger}(\mathbf{a})\,\hat{\mathcal{T}}(\mathbf{a}) &= e^{i\hat{\mathcal{O}}^{\dagger}}e^{-i\hat{\mathcal{O}}} \\ &= e^{i\hat{\mathcal{O}}}e^{-i\hat{\mathcal{O}}} \\ &= e^{i\hat{\mathcal{O}}-i\hat{\mathcal{O}}} \\ &= \hat{1}\end{aligned}$$

where the product of exponentials is $e^{i\hat{\mathcal{O}}}e^{-i\hat{\mathcal{O}}} = e^{i\hat{\mathcal{O}}-i\hat{\mathcal{O}}}$ because $\hat{\mathcal{O}}$ commutes with itself. $\hat{\mathcal{O}}(\mathbf{a})$ is therefore an observable associated with translations.

(Proof: We show a more general result. Suppose two operators commute, $\left[\hat{A},\hat{B}\right] = 0$. Then $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}$. To show this, insert a parameter and consider the derivative of the product, $\hat{C}(\lambda) = e^{i\lambda\hat{A}}e^{-i\lambda\hat{B}}$,

$$\begin{aligned}\frac{d}{d\lambda}\hat{C}(\lambda) &= \frac{d}{d\lambda}\left(e^{i\lambda\hat{A}}e^{i\lambda\hat{B}}\right)\\ &= e^{i\lambda\hat{A}}i\hat{A}e^{i\lambda\hat{B}} + e^{i\lambda\hat{A}}i\hat{B}e^{i\lambda\hat{B}}\\ &= i\left(\hat{A}+\hat{B}\right)e^{i\lambda\hat{A}}e^{i\lambda\hat{B}}\\ &= i\left(\hat{A}+\hat{B}\right)\hat{C}(\lambda)\end{aligned}$$

Since $i\left(\hat{A}+\hat{B}\right)$ is independent of λ, we see that the differential equation is solved by

$$e^{i\lambda\hat{A}}e^{-i\lambda\hat{B}} = \hat{C}(0)\,e^{i\lambda\left(\hat{A}+\hat{B}\right)}$$

Noting that $\hat{C}(0) = \hat{1}$ and setting $\lambda = 1$, we arrive at

$$e^{i\hat{A}}e^{-i\hat{B}} = \hat{C}(0)\,e^{i\left(\hat{A}+\hat{B}\right)}$$

In particular, setting $\hat{A} = \hat{\mathcal{O}}$ and $\hat{B} = -\hat{\mathcal{O}}$,

$$e^{i\hat{\mathcal{O}}}e^{-i\hat{\mathcal{O}}} = \hat{1}$$

This demonstrates the claim.)

From property 2, together with the exponential form, $\hat{\mathcal{T}}^{-1}(\mathbf{a}) = e^{i\hat{\mathcal{O}}(\mathbf{a})}$ we see that

$$\begin{aligned}e^{i\hat{\mathcal{O}}(\mathbf{a})} &= e^{-i\hat{\mathcal{O}}(-\mathbf{a})}\\ \hat{\mathcal{O}}(\mathbf{a}) &= -\hat{\mathcal{O}}(-\mathbf{a})\end{aligned}$$

that is, $\hat{\mathcal{O}}$ is an odd function of $\mathbf{a}$.

4.2 Momentum as the generator of infinitesimal translations

Now consider an infinitesimal translation, $\mathbf{a} = d\mathbf{x}$. Since $\hat{\mathcal{T}}(\mathbf{a})$ depends continuously on $\mathbf{a}$, $\hat{\mathcal{T}}(d\mathbf{x})$ must differ infinitesimally from the identity operator,

$$\hat{\mathcal{T}}(d\mathbf{x}) = \hat{1} - id\mathbf{x}\cdot\hat{\mathbf{K}}$$

where the components of $\hat{\mathbf{K}}$ are the generators of infintesimal translations. From property 1 above,

$$\begin{aligned}0 &= \left[\hat{\mathcal{T}}(\mathbf{a}),\hat{\mathcal{T}}(\mathbf{b})\right]\\ &= \left[\hat{1} - id\mathbf{a}\cdot\hat{\mathbf{K}}, \hat{1} - id\mathbf{b}\cdot\hat{\mathbf{K}}\right]\\ &= -da^i db^j\left[\hat{K}_i,\hat{K}_j\right]\end{aligned}$$

and since the infinitesimal displacements are arbitrary, the $\hat{K}_i$ commute with one another, $\left[\hat{K}_i,\hat{K}_j\right] = 0$.

We examine the effect of an infinitesimal translation by a fixed Δx on a general state, $|\psi\rangle$,

$$\left(\hat{1} - i\Delta x\cdot\hat{K}_x\right)|\psi\rangle = \hat{\mathcal{T}}(\Delta x)\,|\psi\rangle$$

$$
\begin{aligned}
&= \int dx' \, \hat{\mathcal{T}}(\Delta x) \left| x' \right\rangle \left\langle x' \mid \psi \right\rangle \\
&= \int dx' \, \left| x' + \Delta x \right\rangle \left\langle x' \mid \psi \right\rangle \\
&= \int dx'' \, \left| x'' \right\rangle \left\langle x'' - \Delta x \mid \psi \right\rangle
\end{aligned}
$$

where we have set $x'' = x' + \Delta x$. Now expand the wave function in a Taylor series,

$$
\begin{aligned}
\left\langle x'' - \Delta x \mid \psi \right\rangle &= \psi\left(x'' - \Delta x\right) \\
&= \psi\left(x''\right) - \frac{\partial \psi}{\partial x''} \Delta x
\end{aligned}
$$

Then, substituting this for $\left\langle x'' - \Delta x \mid \psi \right\rangle$ and looking at the resulting equation in the x-basis,

$$
\begin{aligned}
\langle x| \left(\hat{1} - i \Delta x \hat{K}_x\right) |\psi\rangle &= \langle x| \int dx'' \, \left| x'' \right\rangle \left(\psi\left(x''\right) - \frac{\partial \psi}{\partial x''} \Delta x\right) \\
\psi(x) - i \Delta x \langle x| \hat{K}_x |\psi\rangle &= \int dx'' \, \delta\left(x'' - x\right) \left(\psi\left(x''\right) - \frac{\partial \psi}{\partial x''} \Delta x\right) \\
\psi(x) - i \Delta x \langle x| \hat{K}_x |\psi\rangle &= \psi(x) - \frac{\partial \psi}{\partial x} \Delta x \\
\langle x| \hat{K}_x |\psi\rangle &= -i \frac{\partial \psi}{\partial x}
\end{aligned}
$$

and we see that the infinitesimal generator of translations is just a multiple of the momentum operator,

$$
\hat{K}_x = \frac{1}{\hbar} \hat{P}_x
$$

The same result holds for the other components of $\hat{\mathbf{K}}$, so we conclude

$$
\hat{\mathbf{K}} = \frac{1}{\hbar} \hat{\mathbf{P}}
$$

The momentum is the generator of infinitesimal translations,

$$
\hat{\mathcal{T}}(d\mathbf{x}) = \hat{1} - \frac{i}{\hbar} d\mathbf{x} \cdot \hat{\mathbf{P}}
$$

5 Commutation with the position operator

The effect of a translation on a position ket is to give a new position ket,

$$
\hat{\mathcal{T}}(\mathbf{a}) |\mathbf{x}\rangle = |\mathbf{x} + \mathbf{a}\rangle
$$

We use this to compute the commutator of the momentum operator with the position operator. For an infinitesmal translation in the direction $d\mathbf{a} = \mathbf{i} dx + \mathbf{j} dy$ with $\hat{X}$, the x-position operator, we have

$$
\begin{aligned}
\left[\hat{X}, \hat{\mathcal{T}}(d\mathbf{a})\right] |\mathbf{x}\rangle &= \left[\hat{X}, \hat{1} - \frac{i}{\hbar} d\mathbf{a} \cdot \hat{\mathbf{P}}\right] |\mathbf{x}\rangle \\
\left(\hat{X} \hat{\mathcal{T}}(d\mathbf{a}) - \hat{\mathcal{T}}(d\mathbf{a}) \hat{X} |\mathbf{x}\rangle\right) |\mathbf{x}\rangle &= -\frac{i}{\hbar} dx \left[\hat{X}, \hat{P}_x\right] |\mathbf{x}\rangle - \frac{i}{\hbar} dy \left[\hat{X}, \hat{P}_y\right] |\mathbf{x}\rangle \\
\hat{X} |\mathbf{x} + d\mathbf{a}\rangle - \hat{\mathcal{T}}(d\mathbf{x}) x |\mathbf{x}\rangle &= -\frac{i}{\hbar} dx \left[\hat{X}, \hat{P}_x\right] |\mathbf{x}\rangle - \frac{i}{\hbar} dy \left[\hat{X}, \hat{P}_y\right] |\mathbf{x}\rangle \\
(x + dx) |\mathbf{x} + d\mathbf{x}\rangle - x |\mathbf{x} + d\mathbf{x}\rangle &= -\frac{i}{\hbar} dx \left[\hat{X}, \hat{P}_x\right] |\mathbf{x}\rangle - \frac{i}{\hbar} dy \left[\hat{X}, \hat{P}_y\right] |\mathbf{x}\rangle \\
dx |\mathbf{x} + d\mathbf{x}\rangle &= -\frac{i}{\hbar} dx \left[\hat{X}, \hat{P}_x\right] |\mathbf{x}\rangle - \frac{i}{\hbar} dy \left[\hat{X}, \hat{P}_y\right] |\mathbf{x}\rangle
\end{aligned}
$$

The ket $|\mathbf{x}+d\mathbf{x}\rangle$ will differ infinitesimally from $|\mathbf{x}\rangle$, so to first order in dx_i, we have $dx\,|\mathbf{x}+d\mathbf{x}\rangle \approx dx\,|\mathbf{x}\rangle$. Then we may treat this as an operator relation. Then since dx and dy are independent, we identify

$$\begin{aligned} dx &= -\frac{i}{\hbar}dx\left[\hat{X},\hat{P}_x\right] \\ 0 &= -\frac{i}{\hbar}dx\left[\hat{X},\hat{P}_y\right] \end{aligned}$$

Therefore, the position operator for the x-direction commutes with the momentum operator in the y-direction, but for the position operator in the same direction as the translation,

$$\left[\hat{X}_i,\hat{P}_j\right] = i\hbar\delta_{ij}\hat{\mathbf{1}}$$

This is true for any pair of directions.

Exercise: Generalize the preceding argument to show that

$$\left[\hat{X}_i,\hat{P}_j\right] = i\delta_{ij}\hat{\mathbf{1}}$$

where $i,j=1,2,3$.

6 Finite translations

Using the infinitesimal form, we may recover the form of a finite translation.

Consider the effect of multiple translations:

$$\begin{aligned} \hat{\mathcal{T}}(\mathbf{a})|\mathbf{x}\rangle &= |\mathbf{x}+\mathbf{a}\rangle \\ \hat{\mathcal{T}}(\mathbf{b})\hat{\mathcal{T}}(\mathbf{a})|\mathbf{x}\rangle &= |(\mathbf{x}+\mathbf{a})+\mathbf{b}\rangle \\ &= |(\mathbf{x}+\mathbf{b})+\mathbf{a}\rangle \\ &= \hat{\mathcal{T}}(\mathbf{a})\hat{\mathcal{T}}(\mathbf{b})|\mathbf{x}\rangle \end{aligned}$$

and furthermore we see that in general,

$$\hat{\mathcal{T}}(\mathbf{a})\hat{\mathcal{T}}(\mathbf{b})|\mathbf{x}\rangle = \hat{\mathcal{T}}(\mathbf{a}+\mathbf{b})|\mathbf{x}\rangle$$

when acting on coordinate basis kets. For an arbitrary state,

$$\begin{aligned} \hat{\mathcal{T}}(\mathbf{a})\hat{\mathcal{T}}(\mathbf{b})|\psi\rangle &= \int d^3x\hat{\mathcal{T}}(\mathbf{a})\hat{\mathcal{T}}(\mathbf{b})|\mathbf{x}\rangle\langle\mathbf{x}\mid\psi\rangle \\ &= \int d^3x\hat{\mathcal{T}}(\mathbf{a}+\mathbf{b})|\mathbf{x}\rangle\langle\mathbf{x}\mid\psi\rangle \\ &= \hat{\mathcal{T}}(\mathbf{a}+\mathbf{b})|\psi\rangle \end{aligned}$$

so the result holds on all states and therefore holds as an operator identity,

$$\hat{\mathcal{T}}(\mathbf{a})\hat{\mathcal{T}}(\mathbf{b}) = \hat{\mathcal{T}}(\mathbf{a}+\mathbf{b})$$

Now, hold two components of the displacement $\mathbf{a}$ fixed and change the third by a small amount. For concreteness, let $a^1 \to a^1+\varepsilon$. Then we have

$$\begin{aligned} \hat{\mathcal{T}}(\mathbf{a}+\varepsilon\mathbf{i}) &= \hat{\mathcal{T}}(\mathbf{a})\hat{\mathcal{T}}(\varepsilon\mathbf{i}) \\ \lim_{\varepsilon\to0}\frac{\hat{\mathcal{T}}(\mathbf{a}+\varepsilon\mathbf{i})-\hat{\mathcal{T}}(\mathbf{a})}{\varepsilon} &= \frac{1}{\varepsilon}\hat{\mathcal{T}}(\mathbf{a})\left(\hat{\mathcal{T}}(\varepsilon\mathbf{i})-1\right) \\ \frac{d}{da^1}\hat{\mathcal{T}}(\mathbf{a}) &= \frac{1}{\varepsilon}\hat{\mathcal{T}}(\mathbf{a})\left(\left(1-\frac{i}{\hbar}\varepsilon\hat{P}_x\right)-1\right) \\ \frac{d}{da^1}\hat{\mathcal{T}}(\mathbf{a}) &= -\frac{i}{\hbar}\hat{\mathcal{T}}(\mathbf{a})\hat{P}_x \end{aligned}$$

This differential equation is satisfied by

$$\hat{\mathcal{T}}(\mathbf{a}) = \hat{A}\left(a^2, a^3\right) e^{-\frac{i}{\hbar}a^1 \hat{P}_x}$$

where $\hat{A}$ is independent of a^1. The same argument holds for a^2 and a^3 as well, and since all the $\hat{P}_i$ it must be the case that

$$\begin{aligned} \hat{\mathcal{T}}(\mathbf{a}) &= \hat{A}_0 e^{-\frac{i}{\hbar}a^1 \hat{P}_1} e^{-\frac{i}{\hbar}a^2 \hat{P}_2} e^{-\frac{i}{\hbar}a^3 \hat{P}_3} \\ &= \hat{A}_0 e^{-\frac{i}{\hbar}\mathbf{a}\cdot\hat{\mathbf{P}}} \end{aligned}$$

for some fixed operator $\hat{A}_0$. Since $\hat{\mathcal{T}}(\mathbf{0})$ is the identity, we must have $\hat{A}_0 = \hat{1}$.

We have shown that a finite translation is found by exponentiating a linear combination of the generators,

$$\hat{\mathcal{T}}(\mathbf{a}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\mathbf{P}}}$$

where $\boldsymbol{a} \cdot \hat{\mathbf{P}}$ is a general element of the Lie algebra.