Change of basis

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For notational simplicity, we consider the cases of finite or countable bases first, then continuum bases.

1 A complete countable basis

We know that the eigenvectors of Hermitian operators may be chosen to form a complete, orthonormal set of basis states. Let

$$S = \{|a_k\rangle \mid k = 1, 2, \ldots, N\}$$

where N may be finite or infinite, be a complete orthonormal basis, with

$$\hat{\mathcal{O}} |a_k\rangle = a_k |a_k\rangle$$

for some Hermitian operator, $\hat{\mathcal{O}}$. We may turn this around to use a complete, orthonormal set to build many diagonal operators:

$$\hat{\mathcal{O}}' \equiv \sum_{k=1}^{N} \alpha_k |a_k\rangle \langle a_k|$$

The complex numbers α_k will be its eigenvalues, and it will have the same eigenvectors as $\hat{\mathcal{O}}$, sinc

$$\begin{aligned} \hat{\mathcal{O}}' |a_m\rangle &= \sum_{k=1}^{N} \alpha_k |a_k\rangle \langle a_k \mid a_m\rangle \\ &= \sum_{k=1}^{N} \alpha_k |a_k\rangle \delta_{km} \\ &= \alpha_m |a_m\rangle \end{aligned}$$

By choosing α_m real for each a_m, we insure that $\hat{\mathcal{O}}'$ is Hermitian. Therefore, we may find many Hermitian operators with a given complete orthonormal set of eigenstates. All such operators commute with one another,

$$\begin{aligned} \left[\hat{\mathcal{O}}', \hat{\mathcal{O}}''\right] &= \left[\left(\sum_{k=1}^{N} \alpha_k |a_k\rangle \langle a_k|\right), \left(\sum_{m=1}^{N} \beta_m |a_m\rangle \langle a_m|\right)\right] \\ &= \sum_{k,m=1}^{N} \alpha_k \beta_m |a_k\rangle \langle a_k \mid a_m\rangle \langle a_m| - \sum_{a_k \in K} \sum_{a_m \in K} \beta_m \alpha_k |a_m\rangle \langle a_m \mid a_k\rangle \langle a_k| \\ &= \sum_{k,m=1}^{N} \left(\alpha_k \beta_m \delta_{km} |a_k\rangle \langle a_m| - \beta_m \alpha_k \delta_{km} |a_m\rangle \langle a_k|\right) \\ &= \sum_{m=1}^{N} \left(\alpha_m \beta_m - \beta_m \alpha_m\right) |a_m\rangle \langle a_m| \\ &= 0 \end{aligned}$$

2 Change of basis

If we have any quantum state, it may be expanded in term of any complete basis:

$$\begin{aligned} |\psi\rangle &= \hat{1}\,|\psi\rangle \\ &= \sum_{k=1}^{N} |a_k\rangle \langle a_k \mid \psi\rangle \end{aligned}$$

so the complex numbers $\langle a_k | \psi\rangle$ represent the *state* $|\psi\rangle$ in the *basis* $|a_k\rangle$. We would like to be able to change the basis without changing the state.

To change the basis, we need the relationship between two bases, $|a_k\rangle$ and $|b_k\rangle$. Because these span the same vector space, there are equal numbers, N, of indices,

$$\begin{aligned} a_k &\in \{a_1, a_2, \cdots, a_N\} \\ b_k &\in \{b_1, b_2, \cdots, b_N\} \end{aligned}$$

Ordering these (arbitrarily, but definitely), puts them in 1-1 correspondence, $a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_2, \ldots, a_N \leftrightarrow b_N$. In general, let $a_m \leftrightarrow b_m$. Now, since any one of the basis kets $|b_m\rangle$ is also a state, it may be expanded in terms of the $|a_k\rangle$,

$$|b_m\rangle = \sum_{k=1}^{N} |a_k\rangle \langle a_k \mid b_m\rangle$$

We may view the numbers $\langle a_k \mid b_m\rangle$ as components of an $N \times N$ matrix $\hat{U}$, which acts on the basis $|a_k\rangle$ to give $|b_m\rangle$. We construct the matrix in the $|a_k\rangle$ basis by multiplying $|a_m\rangle \langle a_n|$ by the mn component $\langle a_m \mid b_n\rangle$,

$$\begin{aligned} \hat{U} &= \sum_{m,n} U_{mn} |a_m\rangle \langle a_n| \\ &= \sum_{m,n} \langle a_m \mid b_n\rangle |a_m\rangle \langle a_n| \\ &= \sum_{m,n} |a_m\rangle \langle a_m \mid b_n\rangle \langle a_n| \\ &= \sum_{n} 1\,|b_n\rangle \langle a_n| \end{aligned}$$

Thus, the matrix is the sum over corresponding pairs,

$$\hat{U} = \sum_{m} |b_m\rangle \langle a_m|$$

Checking the action of $\hat{U}$ on any $|a_k\rangle$,

$$\begin{aligned} \hat{U}\,|a_k\rangle &= \sum_{m} |b_m\rangle \langle a_m \mid a_k\rangle \\ &= \sum_{m} \delta_{mk} |b_m\rangle \\ &= |b_k\rangle \end{aligned}$$

we get the corresponding $|b_k\rangle$, as desired.

We easily prove that $\hat{U}$ is unitary, since, with $\hat{U}^\dagger = \sum_m |a_m\rangle\langle b_m|$, we have

$$\begin{aligned}
\hat{U}^\dagger \hat{U} &= \left(\sum_m |a_m\rangle\langle b_m|\right)\left(\sum_n |b_n\rangle\langle a_n|\right) \\
&= \sum_{m,n} |a_m\rangle\langle b_m \mid b_n\rangle\langle a_n| \\
&= \sum_{m,n} \delta_{mn} |a_m\rangle\langle a_n| \\
&= \sum_m |a_m\rangle\langle a_m| \\
&= \hat{1}
\end{aligned}$$

3 Continuum bases

The bra-ket notation applies equally well to both discrete and continuum bases, the only major changes being that sums are replaced by integrals and Kronecker deltas become Dirac deltas. Here is a full set of correspondences:

State	$\|\chi\rangle$	$\|\psi\rangle$
Basis	$\|a_k\rangle$	$\|\xi\rangle$
Eigenstate	$\hat{\mathcal{O}}\|a_k\rangle = a_k\|a_k\rangle$	$\hat{\mathcal{O}}\|\xi\rangle = \xi\|\xi\rangle$
Identity (completeness)	$\hat{1} = \sum_k \|a_k\rangle\langle a_k\|$	$\hat{1} = \int d\xi\, \|\xi\rangle\langle\xi\|$
Orthonormality	$\langle a_m \mid a_n\rangle = \delta_{mn}$	$\langle \xi' \mid \xi''\rangle = \delta(\xi' - \xi'')$
State in basis	$\|\chi\rangle = \sum_k \|a_k\rangle\langle a_k \mid \chi\rangle$	$\|\psi\rangle = \int d\xi\, \|\xi\rangle\langle\xi \mid \psi\rangle$
Defining operator	$\hat{\mathcal{O}} = \sum_k a_k \|a_k\rangle\langle a_k\|$	$\hat{\mathcal{O}} = \int d\xi\, \xi\, \|\xi\rangle\langle\xi\|$
General operator	$\hat{\mathcal{A}} = \sum_{k,m} \alpha_{km} \|a_k\rangle\langle a_m\|$	$\hat{\mathcal{A}} = \int d\xi \int d\xi'\, A(\xi,\xi')\, \|\xi\rangle\langle\xi'\|$
Matrix element	$\left[\hat{\mathcal{O}}\right]_{km} = \langle a_k\| \hat{\mathcal{O}} \|a_m\rangle$	$\hat{\mathcal{O}}(\xi',\xi'') = \langle\xi'\| \hat{\mathcal{O}} \|\xi''\rangle$

In a continuous basis, the expansion coefficients of a state is a *function*,

$$\langle \xi \mid \psi\rangle = \psi(\xi)$$

and the bra-ket notation allows us to talk about the state as a vector, without choosing a position, momentum, or some other basis for the vectors.

The eigenvectors of any Hermetian operator still form a complete, orthonormal basis,

$$\hat{O}|\lambda\rangle = \lambda|\lambda\rangle$$

and Hermitian operators correspond to dynamical variables – the phyical quantitites we wish to measure. This lets us define bases of position, momentum or energy eigenkets, though many more are possible.

4 Change of basis

Change of a continuous basis, $|\xi\rangle \to |\chi\rangle$, follows the same pattern as for a countable basis, but with the sum $\hat{U} = \sum_m |b_m\rangle\langle a_m|$ replaced by an integral and the 1-1 correspondence $a_m \longleftrightarrow b_m$ replaced by a monotonic function $\xi \longleftrightarrow \chi(\xi)$. In order for the transformation to be uniform, we require this function to be direct identification, $\frac{d\chi}{d\xi} = 1$. This condition is needed to insure unitarity.

$$\hat{U} = \int d\xi\, |\chi(\xi)\rangle\langle\xi|$$

This does just what we require, for if $|\xi_0\rangle$ is any particular eigenket then

$$\begin{aligned}\hat{U}|\xi_0\rangle &= \int d\xi\, |\chi(\xi)\rangle \langle \xi \mid \xi_0\rangle \\ &= \int d\xi\, |\chi(\xi)\rangle\, \delta(\xi - \xi_0) \\ &= |\chi(\xi_0)\rangle\end{aligned}$$

so that $\hat{U}$ gives the corresponding ket $|\chi(\xi_0)\rangle$ in the χ basis.

Unitarity follows in the same way as before. With

$$\hat{U}^\dagger = \int d\xi\, |\xi\rangle \langle \chi(\xi)|$$

we have

$$\begin{aligned}\hat{U}^\dagger \hat{U} &= \int d\xi'\, |\xi'\rangle \langle \chi(\xi')| \int d\xi\, |\chi(\xi)\rangle \langle \xi| \\ &= \int d\xi' \int d\xi\, |\xi'\rangle \langle \chi(\xi') \mid \chi(\xi)\rangle \langle \xi| \\ &= \int d\xi' \int d\xi\, |\xi'\rangle\, \delta(\chi(\xi') - \chi(\xi)) \langle \xi|\end{aligned}$$

Now change variable. Letting $\xi(\chi)$ be the inverse function, $\xi(\chi) = \chi$,

$$d\xi = \frac{d\xi}{d\chi} d\chi = d\chi$$

so that

$$\begin{aligned}\hat{U}^\dagger \hat{U} &= \int d\chi' \int d\chi\, |\xi'\rangle\, \delta(\chi(\xi') - \chi(\xi)) \langle \xi| \\ &= \int d\chi' \int d\chi \delta(\chi' - \chi)\, |\xi'(\chi')\rangle \langle \xi(\chi)| \\ &= \int |\xi'(\chi)\rangle \langle \xi(\chi)|\, d\chi \\ &= \int |\xi\rangle \langle \xi|\, d\xi \\ &= 1\end{aligned}$$

Notice that if we allowed a more general function $\chi(\xi)$, then we would have $d\xi = \frac{d\xi}{d\chi} d\chi$ and we would not get the identity here, but instead a more general diagonal operator,

$$\hat{U}^\dagger \hat{U} = \int \frac{d\xi}{d\chi}\, |\xi\rangle \langle \xi|\, d\xi$$

We now turn to the examples of the position and momentum bases and their relationship to one another.

5 Position basis

The most familiar basis is the position basis. Since we can measure position vectors, we have a vector of Hermitian position operators, $\hat{\mathbf{X}} = \left(\hat{X}, \hat{Y}, \hat{Z}\right) \Leftrightarrow \hat{X}_i$, $\hat{\mathbf{X}}^\dagger = \hat{\mathbf{X}}$, with continuous real eigenvalues, $\mathbf{x}$, giving the position vector of a particle:

$$\hat{\mathbf{X}}\, |\mathbf{x}\rangle = \mathbf{x}\, |\mathbf{x}\rangle$$

When expanded in the position basis, the components comprise the familiar wave function. To expand a state $|\psi\rangle$ in a position basis, we use the position eigenbra, $\langle\mathbf{x}|$, to get

$$\psi(\mathbf{x}) = \langle\mathbf{x} \mid \psi\rangle$$

This is just a complex number for each position vector $\mathbf{x}$, i.e., a function, $\psi(\mathbf{x})$. Normalization of the wave function may be written as

$$\begin{aligned} 1 &= \langle\psi \mid \psi\rangle \\ &= \langle\psi|\,\hat{1}\,|\psi\rangle \\ &= \int d^3x\,\langle\psi \mid \mathbf{x}\rangle\langle\mathbf{x} \mid \psi\rangle \\ &= \int d^3x\;\psi^*(\mathbf{x})\,\psi(\mathbf{x}) \end{aligned}$$

so we recover our usual rule with $\psi^*(\mathbf{x})\,\psi(\mathbf{x})$ as the probability density.

Measurements in the x, y and z directions commute:

$$\left[\hat{X}_i, \hat{X}_j\right] = 0$$

so that all three eigenvalues may be specified at once. Eigenkets are labelled by all three corresponding simultaneous eigenvalues $|\mathbf{x}\rangle = |x, y, z\rangle$.

6 Commutation and simultaneous eigenkets

The correspondence between commutation and the existence of simultaneous eigenvalues is a general property. Suppose we have a set of basis states which are simultaneous eigenkets, $|\alpha, \beta\rangle$ of two different Hermitian dynamical variables $\hat{A}, \hat{B}$, that is

$$\begin{aligned} \hat{A}\,|\alpha,\beta\rangle &= \alpha\,|\alpha,\beta\rangle \\ \hat{B}\,|\alpha,\beta\rangle &= \beta\,|\alpha,\beta\rangle \end{aligned}$$

Then the commutator acting on the basis gives

$$\begin{aligned} \left[\hat{A},\hat{B}\right]|\alpha,\beta\rangle &= \left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)|\alpha,\beta\rangle \\ &= \hat{A}\hat{B}\,|\alpha,\beta\rangle - \hat{B}\hat{A}\,|\alpha,\beta\rangle \\ &= \hat{A}\beta\,|\alpha,\beta\rangle - \hat{B}\alpha\,|\alpha,\beta\rangle \\ &= \beta\hat{A}\,|\alpha,\beta\rangle - \alpha\hat{B}\,|\alpha,\beta\rangle \\ &= (\beta\alpha - \alpha\beta)\,|\alpha,\beta\rangle \\ &= 0 \end{aligned}$$

so the action of the commutator *on any basis ket* gives zero. Therefore, if the basis is complete, we may write *any* state as a superposition,

$$|\psi\rangle = \iint d\alpha d\beta\; f(\alpha,\beta)\,|\alpha,\beta\rangle$$

Then we have

$$\begin{aligned} \left[\hat{A},\hat{B}\right]|\psi\rangle &= \left[\hat{A},\hat{B}\right]\iint d\alpha d\beta\, f(\alpha,\beta)\,|\alpha,\beta\rangle \\ &= \iint d\alpha d\beta A(\alpha,\beta)\left[\hat{A},\hat{B}\right]|\alpha,\beta\rangle \\ &= 0 \end{aligned}$$

Since $\left[\hat{A},\hat{B}\right]$ vanishes acting on *every state*, it is the zero operator,

$$\left[\hat{A},\hat{B}\right]=0$$

Conversely, suppose $\left[\hat{A},\hat{B}\right]$ vanishes on every ket. Then starting with the eigenkets of $\hat{A}$,

$$\hat{A}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$$

we have

$$\begin{aligned}\left[\hat{A},\hat{B}\right]\left|\alpha\right\rangle &= 0\\ \hat{A}\hat{B}\left|\alpha\right\rangle-\hat{B}\hat{A}\left|\alpha\right\rangle &= 0\\ \hat{A}\hat{B}\left|\alpha\right\rangle-\alpha\hat{B}\left|\alpha\right\rangle &= 0\end{aligned}$$

Defining $\left|\lambda\right\rangle\equiv\hat{B}\left|\alpha\right\rangle$ this shows that

$$\hat{A}\left|\lambda\right\rangle-\alpha\left|\lambda\right\rangle=0$$

so that $\hat{B}\left|\alpha\right\rangle$ is also an eigenket of $\hat{A}$ with the eigenvalue α. If $\hat{A}$ is non-degenerate, then we must therefore have $\hat{B}\left|\alpha\right\rangle=\beta\left|\alpha\right\rangle$ for some complex number β. This shows that $\left|\alpha\right\rangle$ is a eigenket of $\hat{B}$ as well, with eigenvalue β, so we may label the states with both eigenvalues, $\left|\alpha,\beta\right\rangle$.

Therefore, there exists a basis of simultaneous eigenstates of two operators if and only if those operators commute. This gives us, at least in principle, a way to label states. If we can find a maximal set of mutually commuting operators, then we may label all states by their eigenvalues.

7 Momentum basis

We also know that momentum is a dynamical variable (an "observable"). Therefore, there is a Hermitian operator

$$\begin{aligned}\hat{\mathbf{P}} &= \hat{\mathbf{P}}^{\dagger}\\ \hat{\mathbf{P}} &= \left(\hat{P}_x,\hat{P}_y,\hat{P}_z\right)=\hat{P}_i\end{aligned}$$

with eigenstates, $\left|\mathbf{p}\right\rangle$, satisfying

$$\hat{\mathbf{P}}\left|\mathbf{p}\right\rangle=\mathbf{p}\left|\mathbf{p}\right\rangle$$

We would like to find the form of this operator in a position basis, that is, we seek the matrix elements $\left\langle\mathbf{x}'\right|\hat{\mathbf{P}}\left|\mathbf{x}"\right\rangle$.

We begin by letting the momentum operator $\hat{\mathbf{P}}$ act on a plane wave state, $\left|\psi\right\rangle$, of wave number $\mathbf{k}$. In a coordinate basis, this satisfies $\left\langle\mathbf{x}\mid\psi\right\rangle=Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Letting $\hat{\mathbf{P}}$ act and inserting an identity operator,

$$\begin{aligned}\hat{\mathbf{P}}\left|\psi\right\rangle &= \hat{\mathbf{P}}\left(\int d^3x\left|\mathbf{x}\right\rangle\left\langle\mathbf{x}\right|\right)\left|\psi\right\rangle\\ &= \hat{\mathbf{P}}\int d^3x\left|\mathbf{x}\right\rangle\left\langle\mathbf{x}\mid\psi\right\rangle\\ &= \hat{\mathbf{P}}\int d^3x\ Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\left|\mathbf{x}\right\rangle\end{aligned}$$

We know that plane waves are eigenstates of the momentum operator, so the action of $\hat{\mathbf{P}}$ must be the momentum, $\hat{\mathbf{P}}\left|\psi\right\rangle=\mathbf{p}\left|\psi\right\rangle=\hbar\mathbf{k}\left|\psi\right\rangle$, and we may write

$$\hbar\mathbf{k}\left|\psi\right\rangle = \int d^3x\ Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\hat{\mathbf{P}}\left|\mathbf{x}\right\rangle$$

Now look project this vector equation into the position basis by taking the inner product with $\langle \mathbf{x}'|$. Placing the $\langle \mathbf{x}'|$ bra on the left,

$$\begin{aligned}
\langle \mathbf{x}'| \hbar\mathbf{k} |\psi\rangle &= \int d^3x\ Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}\rangle \\
\hbar\mathbf{k}\psi(\mathbf{x}') &= \int d^3x\ Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}\rangle \\
\hbar\mathbf{k}Ae^{i\left(\mathbf{k}\cdot\mathbf{x}'-\omega t\right)} &= \int d^3x\ Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}\rangle \\
\hbar\mathbf{k}Ae^{i\mathbf{k}\cdot\mathbf{x}'} &= \int d^3x\ Ae^{i\mathbf{k}\cdot\mathbf{x}} \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}\rangle
\end{aligned}$$

The integral on the right is just the Fourier transform of the matrix element we want. All we need to do is take the inverse transform of both sides. Multiply by $e^{-i\mathbf{k}\cdot\mathbf{x}''}$ and integrate over d^3k,

$$\begin{aligned}
\int d^3k\ \hbar\mathbf{k}Ae^{i\mathbf{k}\cdot\mathbf{x}'}e^{-i\mathbf{k}\cdot\mathbf{x}''} &= \int d^3k \int d^3x\ Ae^{i\mathbf{k}\cdot\left(\mathbf{x}-\mathbf{x}''\right)} \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}\rangle \\
\int d^3k\ \hbar\mathbf{k}e^{i\mathbf{k}\cdot\left(\mathbf{x}'-\mathbf{x}''\right)} &= \int d^3x\ \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}\rangle (2\pi)^3 \delta^3(\mathbf{x}'-\mathbf{x}'') \\
\int d^3k\ \hbar\mathbf{k}e^{i\mathbf{k}\cdot\left(\mathbf{x}'-\mathbf{x}''\right)} &= (2\pi)^3 \langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}''\rangle
\end{aligned}$$

We may evaluate the left hand side using a derivative,

$$\begin{aligned}
\int d^3k\ \hbar\mathbf{k}e^{i\mathbf{k}\cdot\left(\mathbf{x}'-\mathbf{x}"\right)} &= \int d^3k\ \hbar\left(i\boldsymbol{\nabla}_{x''}\right) e^{i\mathbf{k}\cdot\left(\mathbf{x}'-\mathbf{x}''\right)} \\
&= \hbar\left(i\boldsymbol{\nabla}_{''}\right) \int d^3k\ e^{i\mathbf{k}\cdot\left(\mathbf{x}'-\mathbf{x}''\right)} \\
&= (2\pi)^3\, i\hbar\boldsymbol{\nabla}_{x''}\delta^3(\mathbf{x}'-\mathbf{x}'')
\end{aligned}$$

Equating the two results we have the matrix elements of the momentum operator in the position basis:

$$\langle \mathbf{x}'| \hat{\mathbf{P}} |\mathbf{x}''\rangle = i\hbar\boldsymbol{\nabla}_{x''}\delta^3(\mathbf{x}'-\mathbf{x}'')$$

To see how this works, we let $\hat{\mathbf{P}}$ act on a general state, $\hat{\mathbf{P}}|\psi\rangle$ and insert an identity operator,

$$\hat{\mathbf{P}}|\psi\rangle = \hat{\mathbf{P}} \int d^3x' |\mathbf{x}'\rangle \langle \mathbf{x}' \mid \psi\rangle$$

Now, in a position basis,

$$\begin{aligned}
\langle \mathbf{x}| \hat{\mathbf{P}} |\psi\rangle &= \int d^3x' \langle \mathbf{x}| \hat{\mathbf{P}} |\mathbf{x}'\rangle \langle \mathbf{x}' \mid \psi\rangle \\
&= \int d^3x' \left(i\hbar\boldsymbol{\nabla}_{x'}\delta^3(\mathbf{x}-\mathbf{x}')\right) \psi(\mathbf{x}')
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\langle \mathbf{x}| \hat{\mathbf{P}} |\psi\rangle &= \int d^3x' \boldsymbol{\nabla}_{x'} \left(i\hbar\delta^3(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}')\right) - \int d^3x \left(i\hbar\delta^3(\mathbf{x}'-\mathbf{x})\boldsymbol{\nabla}_x\psi(\mathbf{x}')\right) \\
&= -i\hbar \int d^3x' \left(\delta^3(\mathbf{x}-\mathbf{x}')\boldsymbol{\nabla}_{x'}\psi(\mathbf{x}')\right) \\
&= -i\hbar\boldsymbol{\nabla}_x\psi(\mathbf{x})
\end{aligned}$$

where the first term integrates to the boundary and vanishes because the delta function vanishes away from $\mathbf{x}' = \mathbf{x}$. This is the form of the momentum operator which we intuited in deriving the Schrödinger equation.

8 Change of basis: $\langle \mathbf{p}| \longleftrightarrow \langle \mathbf{x}|$

Suppose we are given a state in the momentum basis, $\langle \mathbf{p} \mid \psi \rangle$, and wish to find it in the position basis, $\langle \mathbf{x} \mid \psi \rangle = \psi(\mathbf{x})$. We write the state in the basis we are after and insert the indentity operator in the momentum basis,

$$\begin{aligned} \psi(\mathbf{x}) &= \langle \mathbf{x} \mid \psi \rangle \\ &= \int d^3p \, \langle \mathbf{x} \mid \mathbf{p} \rangle \langle \mathbf{p} \mid \psi \rangle \end{aligned}$$

We need the numbers $\langle \mathbf{x} \mid \mathbf{p} \rangle$ to complete the integral. These are the components of a unitary matrix, since the product of $M = \langle \mathbf{x} \mid \mathbf{p} \rangle$ with its adjoint $M^\dagger = \langle \mathbf{p} \mid \mathbf{x} \rangle$ is

$$\begin{aligned} M^\dagger M &= \int d^3x \, \langle \mathbf{p}' \mid \mathbf{x} \rangle \langle \mathbf{x} \mid \mathbf{p} \rangle \\ &= \langle \mathbf{p}' \mid \mathbf{p} \rangle \\ &= \delta^3(\mathbf{p} - \mathbf{p}') \\ &= \mathbf{1} \end{aligned}$$

$\langle \mathbf{x} \mid \mathbf{p} \rangle$ gives the components of a unitary transformation that takes us from one basis to another. This is important, since it preserves Hermiticity,

$$\hat{\hat{H}} = U \hat{H} U^\dagger$$

implies

$$\begin{aligned} \hat{\hat{H}}^\dagger &= \left(U \hat{H} U^\dagger \right)^\dagger \\ &= U^{\dagger\dagger} \hat{H}^\dagger U^\dagger \\ &= U \hat{H} U^\dagger \end{aligned}$$

The matrix elements $\langle \mathbf{x} \mid \mathbf{p} \rangle$ are found as follows. Projecting $\hat{\mathbf{P}} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle$ with $\langle \mathbf{x}|$, and inserting an identity, we have

$$\begin{aligned} \mathbf{p} \langle \mathbf{x} \mid \mathbf{p} \rangle &= \langle \mathbf{x}| \hat{\mathbf{P}} |\mathbf{p}\rangle \\ &= \int d^3x' \; \langle \mathbf{x}| \hat{\mathbf{P}} |\mathbf{x}'\rangle \langle \mathbf{x}' \mid \mathbf{p} \rangle \\ &= \int d^3x' \; \left(i\hbar \boldsymbol{\nabla}_{x'} \delta^3(\mathbf{x} - \mathbf{x}') \right) \langle \mathbf{x}' \mid \mathbf{p} \rangle \\ &= -i\hbar \int d^3x' \; \delta^3(\mathbf{x} - \mathbf{x}') \, \boldsymbol{\nabla}_{x'} \langle \mathbf{x}' \mid \mathbf{p} \rangle \\ &= -i\hbar \boldsymbol{\nabla}_x \langle \mathbf{x} \mid \mathbf{p} \rangle \end{aligned}$$

This is a simple differential equation for the inner product,

$$\mathbf{p} \langle \mathbf{x} \mid \mathbf{p} \rangle = -i\hbar \boldsymbol{\nabla}_x \langle \mathbf{x} \mid \mathbf{p} \rangle$$

with the solution

$$\langle \mathbf{x} \mid \mathbf{p} \rangle = A e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{x}}$$

The normalization constant A may be found from

$$\begin{aligned}
\delta^3(\mathbf{x}-\mathbf{x}') &= \langle \mathbf{x} \mid \mathbf{x}' \rangle \\
&= \int d^3p \ \langle \mathbf{x} \mid \mathbf{p} \rangle \langle \mathbf{p} \mid \mathbf{x}' \rangle \\
&= \int d^3p \ A e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} A^* e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}'} \\
&= A^* A \int d^3p \ e^{\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= A^* A \hbar^3 \int d^3k \ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= A^* A \hbar^3 (2\pi)^3 \delta^3(\mathbf{x}-\mathbf{x}')
\end{aligned}$$

Choosing the normalization real gives

$$\langle \mathbf{x} \mid \mathbf{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$$

Thus, the transition functions between the position and momentum bases are just Fourier modes.