Quantum Mechanics: Wheeler: Physics 6210

Assignment 1

READ: Start reading Sakurai, Chapter 1.

PROBLEMS:

S.1.1: This is a straightforward rearrangement of terms. First warm up by proving:

\[ [AB, C] = A[B, C] + [A, C]B \]

and the Jacobi identity:

\[ [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \]

S.1.2: Sakurai does not define the Pauli matrices, \( \sigma^i \). They are the usual Pauli matrices (see pg. 164, eq.(3.2.32)):

\[
\sigma_i = (\sigma_1, \sigma_2, \sigma_3)
\]

where

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Also, where he writes \( a_0 \) he really means \( a_0 \mathbf{1} \), where \( \mathbf{1} \) is the unit 2 \times 2 matrix.

S.1.3: Let’s review some facts about determinants. The determinant of a product is the product of the determinants,

\[ \det(AB) = \det(A) \det(B) \]

Now suppose \( B = A^{-1} \), so we have \( AA^{-1} = 1 \). Then

\[
\begin{align*}
\det( AA^{-1} ) &= \det(1) \\
\det(A) \det(A^{-1}) &= 1 \\
\det(A^{-1}) &= \frac{1}{\det(A)}
\end{align*}
\]

Looking at the problem, we see that the result follows if we can show that the matrices \( \exp \left( \frac{i}{2} n \cdot \sigma \right) \) and \( \exp \left( -\frac{i}{2} n \cdot \sigma \right) \) are inverse to one another. We can show this using the Campbell-Baker-Hausdorff theorem, which states that for any two operators, \( A \) and \( B \),

\[ e^A e^B = e^{A+B+[A,B]+...} \]
where "," \ldots ", includes commutators of commutators, \([A, [A, B]],\) of all orders. Therefore, if \(A\) and \(B\) commute, the exponentials add just like numbers. But clearly,

\[
\left[ \frac{i\varphi}{2} \mathbf{n} \cdot \sigma, -\frac{i\varphi}{2} \mathbf{n} \cdot \sigma \right] = 0
\]

since anything commutes with itself, so \(\exp\left(\frac{i\varphi}{2} \mathbf{n} \cdot \sigma\right)\) and \(\exp\left(-\frac{i\varphi}{2} \mathbf{n} \cdot \sigma\right)\)

\[
\exp\left(\frac{i\varphi}{2} \mathbf{n} \cdot \sigma\right) \exp\left(-\frac{i\varphi}{2} \mathbf{n} \cdot \sigma\right) = \exp\left(\frac{i\varphi}{2} \mathbf{n} \cdot \sigma - \frac{i\varphi}{2} \mathbf{n} \cdot \sigma\right) = \exp(0) = 1
\]

There is another, more complicated, way to work the same problem. It’s worth going through because we get a useful identity and a lot of practice. The exponential of a matrix is defined by the power series:

\[
e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n
\]

Therefore, we have

\[
\exp\left(\frac{i\varphi}{2} \mathbf{n} \cdot \sigma\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\varphi}{2}\right)^n (\mathbf{n} \cdot \sigma)^n
\]

To evaluate this, we need powers of the matrix

\[
\mathbf{n} \cdot \sigma = n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3
\]

\[
= n_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix}
\]

You can just square this, but let’s practice with indices. We want

\[
(\mathbf{n} \cdot \sigma)^2 = \sum_{i,j=1}^{3} (n_i \sigma_i)(n_j \sigma_j)
\]

\[
= \sum_{i,j=1}^{3} n_i n_j (\sigma_i \sigma_j)
\]

\[
= \sum_{i,j=1}^{3} n_i n_j \left( \delta_{ij} + i \sum_{k=1}^{3} \varepsilon_{ijk} \sigma_k \right)
\]

\[
= \sum_{i,j=1}^{3} n_i n_j \delta_{ij} + i \sum_{i,j,k=1}^{3} n_i n_j \varepsilon_{ijk} \sigma_k
\]

2
\[\sum_{i=1}^{3} n_i n_i + i \sum_{k=1}^{3} (\vec{n} \times \vec{n})_k \sigma_k \]
\[= (\vec{n} \cdot \vec{n}) + 0 \]
\[= 1 \]

This makes all other powers easy:
\[(n \cdot \sigma)^2 = 1 \]
\[(n \cdot \sigma)^3 = (n \cdot \sigma) \]
\[(n \cdot \sigma)^4 = 1 \]

and in general,
\[(n \cdot \sigma)^{2k} = 1 \]
\[(n \cdot \sigma)^{2k+1} = n \cdot \sigma \]

Returning to our exponential, we may split the sum into even and odd parts,
\[\exp \left( \frac{i \varphi}{2} n \cdot \sigma \right) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left( \frac{i \varphi}{2} \right)^{2k} (n \cdot \sigma)^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left( \frac{i \varphi}{2} \right)^{2k+1} (n \cdot \sigma)^{2k+1} \]
\[= 1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left( \frac{\varphi}{2} \right)^{2k} + i (n \cdot \sigma) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left( \frac{\varphi}{2} \right)^{2k+1} \]
\[= 1 \cos \left( \frac{\varphi}{2} \right) + i (n \cdot \sigma) \sin \left( \frac{\varphi}{2} \right) \]

This is an important identity. Now check the inverse,
\[\exp \left( \frac{i \varphi}{2} n \cdot \sigma \right) \exp \left( -\frac{i \varphi}{2} n \cdot \sigma \right) = \left( 1 \cos \left( \frac{\varphi}{2} \right) + i (n \cdot \sigma) \sin \left( \frac{\varphi}{2} \right) \right) \left( 1 \cos \left( \frac{\varphi}{2} \right) - i (n \cdot \sigma) \sin \left( \frac{\varphi}{2} \right) \right) \]
\[= 1 \cos^2 \left( \frac{\varphi}{2} \right) + i (n \cdot \sigma) \cos \left( \frac{\varphi}{2} \right) \sin \left( \frac{\varphi}{2} \right) - i (n \cdot \sigma) \cos \left( \frac{\varphi}{2} \right) \sin \left( \frac{\varphi}{2} \right) - i^2 (n \cdot \sigma)^2 \]
\[= 1 \left( \cos^2 \left( \frac{\varphi}{2} \right) + \sin^2 \left( \frac{\varphi}{2} \right) \right) \]
\[= 1 \]

so we once again see that these two exponentials are inverse to one another.

S.1.5: Bra - ket practice.
S.1.6: Don’t just show that it works – give a derivation. Set up the eigenket condition and deduce the conditions under which it holds.
S.1.7: Remember that the action of an operator is totally defined by its action on an arbitrary state.
S.1.8: These are important relationships, worth checking, and good practice with the bra-ket notation. After you are done, find the matrix representations of $S_x$, $S_y$, and $S_z$. They should look familiar.
S.1.9: Follow the directions: just expand the state in the $|+>, |->$ basis and impose the required condition. You will get a general expression for the eigenket. In order to match the expression given in the problem, you will need to normalize the ket, and multiply by a phase.

S.1.10: This should be easy. After you solve the problem, recast it in the usual matrix notation.