Part I

Free Field Theory

See previous notes.

Part II

Interactions

Chapter 1

Feynman Path integral

1.1 Time and space translations

Recall from quantum mechanics that the energy and momentum operators must extract E and $\mathbf{p}$ from a plane wave. Remembering this lets us get the signs right:

$$\begin{aligned} \hat{H}e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)} &= Ee^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)} \\ \hat{\mathbf{p}}e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)} &= \mathbf{p}e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}-Et)} \end{aligned}$$

It is clear that this is accomplished if we identify

$$\begin{aligned} \hat{H} &= i\hbar\frac{\partial}{\partial t} \\ \hat{\mathbf{p}} &= -i\hbar\boldsymbol{\nabla} \end{aligned}$$

Another way to remember the signs is to think relativistically, making the operators a 4-vector. Then, using our current convention for the metric, $\eta^{\mu\nu} = diag\,(1,-1,-1,-1)$ we have

$$\begin{aligned} \left(\hat{H},\hat{\mathbf{p}}\right)^{\mu} &= i\hbar\eta^{\mu\nu}\frac{\partial}{\partial x^{\nu}} \\ &= \left(i\hbar\frac{\partial}{\partial t},-i\hbar\frac{\partial}{\partial x^{i}}\right) \end{aligned}$$

Now, to accomplish an infinitesimal time translation, $\hat{U}(\Delta t) = \hat{1} + \Delta t\hat{G}$ we let it act on a general ket,

$$|\psi(t+\Delta t)\rangle = \left(\hat{1}+\Delta t\hat{G}\right)|\psi(t)\rangle$$

Introducing a coordinate basis $\langle\mathbf{x}|$ and expanding $\psi(\mathbf{x},t+\Delta t)$ in a Taylor series,

$$\begin{aligned} \psi(\mathbf{x},t+\Delta t) = \langle\mathbf{x}\,|\psi(t+\Delta t)\rangle &= \langle\mathbf{x}|\left(\hat{1}+\Delta t\hat{G}\right)|\psi(t)\rangle \\ \psi(\mathbf{x},t+\Delta t) &= \langle\mathbf{x}|\left(\hat{1}+\Delta t\hat{G}\right)|\psi(t)\rangle \\ \psi(\mathbf{x},t)+\Delta t\frac{\partial}{\partial t}\psi(\mathbf{x},t) &= \psi(\mathbf{x},t)+\Delta t\,\langle\mathbf{x}|\,\hat{G}\,|\psi(t)\rangle \end{aligned}$$

we may identify

$$\begin{aligned} \langle\mathbf{x}|\,\hat{G}\,|\psi(t)\rangle &= \frac{\partial}{\partial t}\psi(\mathbf{x},t) \\ &= -\frac{i}{\hbar}\hat{H}\psi(\mathbf{x},t) \end{aligned}$$

Therefore, an infinitesimal time translation is given by

$$\hat{U}(\Delta t) = \hat{1} - \frac{i}{\hbar}\hat{H}\Delta t$$

Applying the transformation n times and taking the limit as $n\Delta t = t_f - t_0$,

$$\begin{aligned}
\lim_{n\to\infty}\hat{U}(n\Delta t) &= \\
&= \lim_{n\to\infty}\left(\hat{1} - \frac{i}{\hbar}\hat{H}(t_f - \Delta t)\Delta t\right)\cdots\left(\hat{1} - \frac{i}{\hbar}\hat{H}(t_0 + 2\Delta t)\Delta t\right)\left(\hat{1} - \frac{i}{\hbar}\hat{H}(t_0 + \Delta t)\Delta t\right)\left(\hat{1} - \frac{i}{\hbar}\hat{H}(t_0)\Delta t\right)
\end{aligned}$$

If the Hamiltonian is independent of time, the n^{th} term is just $\lim_{n\to\infty}\left(\hat{1} - \frac{i}{\hbar}\hat{H}(t_k)\Delta t\right)^n$ and we may use the binomial theorem to take the limit to yield

$$\hat{U}(t_f; t_0) = e^{-\frac{i}{\hbar}\hat{H}(t_f - t_0)}$$

If the Hamiltonians at any two times commute, $\left[\hat{H}(t_1), \hat{H}(t_2)\right] = 0$ then in the limit only the linear term in $\Delta t \to dt$ survives and the limiting exponent becomes

$$\lim_{\Delta t\to 0}\left(-\frac{i}{\hbar}\hat{H}(t_f - \Delta t)\Delta t \cdots - \frac{i}{\hbar}\hat{H}(t_0 + 2\Delta t)\Delta t - \frac{i}{\hbar}\hat{H}(t_0 + \Delta t)\Delta t - \frac{i}{\hbar}\hat{H}(t_0)\Delta t\right) = -\frac{i}{\hbar}\int_{t_0}^{t_f}\hat{H}(t)\,dt$$

and with some effort one can show that the general case may be summarized by

$$\hat{U}(t_f; t_0) = Te^{-\frac{i}{\hbar}\int_{t_0}^{t_f}\hat{H}(t)dt}$$

where T represents the time ordered product.

Similarly, we may identify spatial translations by considering the infinitesimal case,

$$\begin{aligned}
\psi(\mathbf{x} + \Delta\mathbf{x}, t) = \langle \mathbf{x}\,\hat{T}(\Delta\mathbf{x})|\psi(t)\rangle &= \langle\mathbf{x}|\left(\hat{1} + \Delta\mathbf{x}\cdot\hat{\mathbf{K}}\right)|\psi(t)\rangle \\
\psi(\mathbf{x}, t) + \Delta\mathbf{x}\cdot\boldsymbol{\nabla}\psi(\mathbf{x}, t) &= \psi(\mathbf{x}, t) + \Delta\mathbf{x}\cdot\langle\mathbf{x}|\,\hat{\mathbf{K}}\,|\psi(t)\rangle
\end{aligned}$$

We therefore identify

$$\begin{aligned}
\langle\mathbf{x}|\,\hat{\mathbf{K}}\,|\psi(t)\rangle &= \boldsymbol{\nabla}\psi(\mathbf{x}, t) \\
&= \frac{i}{\hbar}\hat{\mathbf{P}}\psi(\mathbf{x}, t)
\end{aligned}$$

with the analogous result,

$$\hat{T}(\mathbf{x}_f; \mathbf{x}_0) = e^{\frac{i}{\hbar}\hat{\mathbf{P}}\cdot(\mathbf{x}_f - \mathbf{x}_0)}$$

1.2 Heisenberg and Schrödinger pictures

The Schrödinger wave function places the time dependence of a physical system in the state, $|\psi, t\rangle$, where the state is a vector in Hilbert space that moves in time. This Hilbert space can be described in any basis we choose: coordinate, $|\mathbf{x}\rangle$, momentum $|\mathbf{p}\rangle$, or whatever suits our need.

It is also possible to regard the state as fixed and the basis as changing in time. This is the Heisenberg picture.

1.2.1 Heisenberg operators

Consider an operator acting on a state, then projected onto any other state at time t,

$$\begin{aligned}
\langle\chi, t| \hat{A} |\psi, t\rangle &= \left(\langle\chi, t_0| \hat{U}^\dagger (t, t_0)\right) \hat{A} \left(\hat{U} (t, t_0) |\psi, t_0\rangle\right) \\
&= \langle\chi, t_0| \left(\hat{U}^\dagger (t, t_0) \hat{A}\hat{U} (t, t_0)\right) |\psi, t_0\rangle
\end{aligned}$$

so if we define a time-dependent Heisenberg operator,

$$\hat{A}_H = \hat{A} (t) \equiv \hat{U}^\dagger (t, t_0) \hat{A}_S \hat{U} (t, t_0)$$

then we get the same prediction by looking at $\hat{A} (t)$ acting on the fixed initial state:

$$\langle\chi, t| \hat{A}_S |\psi, t\rangle = \langle\chi, t_0| \hat{A}_H (t) |\psi, t_0\rangle$$

We may replace the Schrodinger equation with evolution equations for operators. Taking the time derivative where $\hat{U} (t, t_0) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$,

$$\begin{aligned}
\frac{d\hat{A}_H}{dt} &= \frac{\partial}{\partial t}\hat{U}^\dagger (t, t_0) \hat{A}_S \hat{U} (t, t_0) + \hat{U}^\dagger (t, t_0) \hat{A}_S \frac{\partial}{\partial t}\hat{U} (t, t_0) \\
&= \frac{i}{\hbar}\hat{H}\hat{U}^\dagger (t, t_0) \hat{A}_S \hat{U} (t, t_0) - \frac{i}{\hbar}\hat{U}^\dagger (t, t_0) \hat{A}_S \hat{U} (t, t_0) \hat{H} \\
&= \frac{i}{\hbar}\hat{H}\hat{A}_H - \frac{i}{\hbar}\hat{A}_H \hat{H}
\end{aligned}$$

and we have the Heisenberg equation of motion,

$$\frac{d\hat{A}_H}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}_H\right]$$

According to Sakurai, this was first written by Dirac.

1.2.2 Heisenberg basis kets

The Heisenberg picture also requires a change in the basis kets. Since basis kets are eigenkets of particular operators, and the operators are now time-depending, the eigenkets also change. We have

$$\hat{A}_S |a\rangle_S = a |a\rangle_S$$

where a state in the Schrödinger basis is given by

$$\psi (a, t) = \langle a |\psi, t\rangle$$

In the Heisenberg picture, these become

$$\begin{aligned}
\hat{A}_H (t) |a, t\rangle_H &= a |a, t\rangle_H \\
\hat{U}^\dagger \hat{A}_S \hat{U} |a, t\rangle_H &= a |a, t\rangle_H \\
\hat{A}_S \hat{U} |a, t\rangle_H &= a\hat{U} |a, t\rangle_H
\end{aligned}$$

so we must have

$$\hat{U} |a, t\rangle_H = |a\rangle_S$$

Inverting,

$$|a, t\rangle_H = \hat{U}^\dagger |a\rangle_S$$

we see that the Heisenberg basis evolves oppositely to the Schrodinger state to give the same result.

1.2.3 Transition amplitudes

Given the time-dependence of the basis kets, we may ask for the probability amplitude for a basis ket $|a, t_0\rangle_H$ at time t_0 to be found in another direction $|b, t_0\rangle_H$ at time t,

$$\langle b, t \,| a, t_0\rangle$$

This is called the *transition amplitude*. For example, the transition amplitude for a system to go from $\mathbf{x}'$ at time t_0 to $\mathbf{x}$ at time t is

$$\langle \mathbf{x}, t \,| \mathbf{x}', t_0\rangle$$

1.3 Propagators

Time evolution in quantum theory is generated by the Hamiltonian operator,

$$\hat{H}\,|\psi, t\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi, t\rangle$$

When the Hamiltonian is independent of time, the time evolution of state is given by

$$|\psi, t\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\,|\psi, t_0\rangle$$

since then taking the derivative,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\,|\psi, t\rangle &= i\hbar\frac{\partial}{\partial t}\left(e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\,|\psi, t_0\rangle\right) \\ &= i\hbar\left(-\frac{i}{\hbar}\hat{H}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\,|\psi, t_0\rangle\right) \\ &= \hat{H}\,|\psi, t\rangle \end{aligned}$$

Inserting an identity in terms of an energy basis, $1 = \sum_a |E_a\rangle\langle E_a|$,

$$\begin{aligned} |\psi, t\rangle &= e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\sum_a |E_a\rangle\langle E_a\,|\psi, t_0\rangle \\ &= \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\,|E_a\rangle\langle E_a\,|\psi, t_0\rangle \end{aligned}$$

Now view the state in a coordinate basis,

$$\begin{aligned} \langle \mathbf{x}\,|\psi, t\rangle &= \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\,\langle \mathbf{x}\,|E_a\rangle\langle E_a\,|\psi, t_0\rangle \\ \psi(\mathbf{x}, t) &= \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\,\langle \mathbf{x}\,|E_a\rangle\langle E_a\,|\psi, t_0\rangle \end{aligned}$$

Inserting one more identity in the coordinate basis, we have

$$\begin{aligned} \psi(\mathbf{x}, t) &= \int d^3x' \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\,\langle \mathbf{x}\,|E_a\rangle\langle E_a\,|\mathbf{x}'\rangle\langle \mathbf{x}'\,|\psi, t_0\rangle \\ &= \int d^3x' \sum_a e^{-\frac{i}{\hbar}E_a(t-t_0)}\,\langle \mathbf{x}\,|E_a\rangle\langle E_a\,|\mathbf{x}'\rangle\,\psi(\mathbf{x}', t_0) \end{aligned}$$

Now define the *propagator*

$$K(\mathbf{x}, t; \mathbf{x}', t_0) \equiv \sum_a \langle \mathbf{x}\,|E_a\rangle\langle E_a\,|\mathbf{x}'\rangle\,e^{-\frac{i}{\hbar}E_a(t-t_0)}$$

so that we have

$$\psi(\mathbf{x},t) = \int d^3x' K(\mathbf{x},t;\mathbf{x}',t_0)\,\psi(\mathbf{x}',t_0)$$

Identifying the propagator for a given problem separates the initial wave function from the potential, allowing a formal solution for the wave function at a later time and arbitrary position. Holding $(\mathbf{x}',t_0)$ fixed, $u_a(x)$ is the stationary state wave function, and $e^{-\frac{i}{\hbar}E_a t}$ is its time dependence, so $K(\mathbf{x},t;\mathbf{x}',t_0)$ satisfies the time-dependent Schrödinger equation. Also,

$$\lim_{t\to t_0} K(\mathbf{x},t;\mathbf{x}',t_0) = \delta^3(\mathbf{x}-\mathbf{x}')$$

The propagator is essentially a Green's function that includes the time evolution, giving the probability amplitude for a particle initially at $\mathbf{x}'$ at t_0 to be found at $\mathbf{x}$ at the later time t. In this way, the propagator is the *transition amplitude* for the system. We can make this explicit:

$$\begin{aligned} K(\mathbf{x},t;\mathbf{x}',t_0) &\equiv \sum_a \langle \mathbf{x}|E_a\rangle\langle E_a|\mathbf{x}'\rangle e^{-\frac{i}{\hbar}E_a(t-t_0)} \\ &= \langle\mathbf{x}|e^{-\frac{i}{\hbar}\hat{H}t}\sum_a |E_a\rangle\langle E_a| e^{\frac{i}{\hbar}\hat{H}t_0}|\mathbf{x}'\rangle \\ &= \langle\mathbf{x}|\hat{U}(t,0)\,\hat{U}^\dagger(t_0,0)|\mathbf{x}'\rangle \end{aligned}$$

so removing the identity $1 = \sum_a |E_a\rangle\langle E_a|$ and identifying the Heisenberg basis states, $\hat{U}^\dagger(t_0,0)|\mathbf{x}'\rangle = |\mathbf{x}',t_0\rangle_H$ and $\hat{U}^\dagger(t,0)|\mathbf{x}\rangle = |\mathbf{x},t\rangle_H$ we have the transition amplitude:

$$K(\mathbf{x},t;\mathbf{x}',t_0) = \langle\mathbf{x},t|\mathbf{x}',t_0\rangle$$

Transition amplitudes, or propagators, have a *composition property.* If we insert the identity operator in the form

$$1 = \int d^3x''\,|\mathbf{x}'',t_1\rangle\langle\mathbf{x}'',t_1|$$

where $t_0 < t_1 < t$, into the transition amplitude, it becomes an integral over a product of transition amplitudes:

$$\langle\mathbf{x},t|\mathbf{x}',t_0\rangle = \int d^3x''\,\langle\mathbf{x},t|\mathbf{x}'',t_1\rangle\langle\mathbf{x}'',t_1|\mathbf{x}',t_0\rangle$$

This shows that the probability amplitude for going from $(\mathbf{x}',t_0)$ to $(\mathbf{x},t)$ is the product of the probability amplitudes for going from $(\mathbf{x}',t_0)$ to an intermediate state at time t_1 and the probability of going from that state to $(\mathbf{x},t)$, summed over all possible intermediate positions. This is just like the composition of conditional probabilities:

$$P_{A\,given\,B} = \sum_C P_{A\,given\,C}P_{C\,given\,B}$$

but it is significant that it applies to probability *amplitudes* instead of probabilities. This fact underlies Bell's theorem.

1.4 The Feynman path integral

We consider a particle with Hamiltonian of the form $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$. This assumption is only important when we carry out the Gaussian integrations in momentum space.

Applying the composition property $N-1$ times in going from (x_0,t_0) to (x_N,t_N),

$$\langle\mathbf{x}_N,t_N|\mathbf{x}_0,t_0\rangle = \int\cdots\int\prod_{i=1}^{N-1} d^3x_i\,\langle\mathbf{x}_N,t_N|\mathbf{x}_{N-1},t_{N-1}\rangle\cdots\langle\mathbf{x}_1,t_1|\mathbf{x}_0,t_0\rangle$$

Now look at a generic intermediate transition amplitude, $\langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle$. To make these kets comparable, we insert time translation and space translation operators:

$$\begin{aligned} \hat{U}(t_i \to t_{i+1}) &= e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} \\ \hat{T}(\mathbf{x}_i \to \mathbf{x}_{i+1}) &= e^{\frac{i}{\hbar}\hat{\mathbf{p}}\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)} \end{aligned}$$

With these we may write

$$\begin{aligned} |\mathbf{x}_{i+i}, t_{i+1}\rangle &= e^{\frac{i}{\hbar}\hat{\mathbf{p}}\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)} e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} |\mathbf{x}_i, t_i\rangle \\ \langle \mathbf{x}_{i+i}, t_{i+1}| &= \langle \mathbf{x}_i, t_i| \, e^{-\frac{i}{\hbar}\hat{\mathbf{p}}\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)} e^{\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} \end{aligned}$$

Now let N be sufficiently large that $t_{i+1} - t_i = \Delta t$ becomes infinitesimal. Looking at an individual term in the product, we use the time translation operator to make the times equal, then expand to first order,

$$\begin{aligned} \langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \langle \mathbf{x}_{i+1}, t_i | \, e^{-\frac{i}{\hbar}\hat{H}(t_{i+1}-t_i)} |\mathbf{x}_i, t_i\rangle \\ &= \langle \mathbf{x}_{i+1}, t_i | \left(1 - \frac{i}{\hbar}\hat{H}\Delta t\right) |\mathbf{x}_i, t_i\rangle \\ &= \langle \mathbf{x}_{i+1}, t_i | \left(1 - \frac{i}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}\Delta t - \frac{i}{\hbar}V(\hat{\mathbf{x}})\,\Delta t\right) |\mathbf{x}_i, t_i\rangle \end{aligned}$$

where $\hat{\mathbf{p}}_i$ is the momentum operator. The potential operator, $\hat{V} = V(\hat{\mathbf{x}})$, acting to the right on $|\mathbf{x}_i, t_i\rangle$ immediately gives $V(\mathbf{x}_i)$, but to evaluate the momentum operator in the Hamiltonian, we insert a momentum basis, $1 = \int d^3p_i \, |\mathbf{p}_i, t_i\rangle \langle \mathbf{p}_i, t_i|$,

$$\begin{aligned} \langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i\rangle \langle \mathbf{p}_i, t_i| \left(1 - \frac{i}{\hbar}\frac{\hat{\mathbf{p}}^2}{2m}\Delta t - \frac{i}{\hbar}V(\mathbf{x}_i)\,\Delta t\right) |\mathbf{x}_i, t_i\rangle \\ &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i\rangle \left(\langle \mathbf{p}_i, t_i \, | \mathbf{x}_i, t_i\rangle - \frac{i\Delta t}{2m\hbar}\langle \mathbf{p}_i, t_i \; \hat{\mathbf{p}}^2 \, |\mathbf{x}_i, t_i\rangle - \frac{i}{\hbar}\langle \mathbf{p}_i, t_i \, |\mathbf{x}_i, t_i\rangle V(\hat{\mathbf{x}}_i)\,\Delta t\right) \\ &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i\rangle \left(1 - \frac{i\mathbf{p}_i^2\Delta t}{2m\hbar} - \frac{i}{\hbar}V(\hat{\mathbf{x}}_i)\,\Delta t\right) \langle \mathbf{p}_i, t_i \, |\mathbf{x}_i, t_i\rangle \end{aligned}$$

Since Δt is tending to zero, we may reassemble the Hamiltonian where we once had the Hamiltonian operator,

$$1 - \frac{i\mathbf{p}_i^2\Delta t}{2m\hbar} - \frac{i}{\hbar}V(\hat{\mathbf{x}}_i)\,\Delta t \quad \approx \quad e^{-\frac{i}{\hbar}H(\mathbf{p}_i,\mathbf{x}_i)}$$

with the expression becoming exact as $\Delta t \to 0$. With this, and using the basis transformation,

$$\langle \mathbf{p}_i, t_i \, |\mathbf{x}_i, t_i\rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}_i}$$

the infinitesimal transition amplitude becomes

$$\begin{aligned} \langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \int d^3p_i \, \langle \mathbf{x}_{i+1}, t_i \, | \mathbf{p}_i, t_i\rangle \langle \mathbf{p}_i, t_i \, |\mathbf{x}_i, t_i\rangle \, e^{-\frac{i\Delta t}{\hbar}H(\mathbf{p}_i,\mathbf{x}_i)} \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3p_i e^{\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}_{i+1}} e^{-\frac{i}{\hbar}\mathbf{p}_i\cdot\mathbf{x}_i} e^{-\frac{i\Delta t}{\hbar}H(\mathbf{p}_i,\mathbf{x}_i)} \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3p_i e^{\frac{i}{\hbar}\mathbf{p}_i\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i)} e^{-\frac{i\Delta t}{\hbar}H(\mathbf{p}_i,\mathbf{x}_i)} \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}\left[\mathbf{p}_i\cdot(\mathbf{x}_{i+1}-\mathbf{x}_i) - H(\mathbf{p}_i,\mathbf{x}_i)\,\Delta t\right] \end{aligned}$$

Finally, taking the limit as $\Delta t \to dt$,

$$\begin{aligned}
\langle \mathbf{x}_{i+i}, t_{i+1} | \, \mathbf{x}_i, t_i \rangle &= \frac{1}{(2\pi\hbar)^3} \lim_{\Delta t \to dt} \int d^3 p_i \exp \frac{i}{\hbar} \left[\mathbf{p}_i \cdot \frac{(\mathbf{x}_{i+1} - \mathbf{x}_i)}{\Delta t} - H(\mathbf{p}_i, \mathbf{x}_i) \right] \Delta t \\
&= \frac{1}{(2\pi\hbar)^3} \int d^3 p_i \exp \frac{i}{\hbar} \left[\mathbf{p}_i \cdot \frac{d\mathbf{x}_i}{dt} - H(\mathbf{p}_i, \mathbf{x}_i) \right] dt \\
&= \frac{1}{(2\pi\hbar)^3} \int d^3 p_i \exp \frac{i}{\hbar} L(\mathbf{p}_i, \mathbf{x}_i) \, dt
\end{aligned}$$

where we recognize the Lagrangian,

$$L(\mathbf{p}_i, \mathbf{x}_i) = \mathbf{p}_i \cdot \dot{\mathbf{x}}_i - H(\mathbf{p}_i, \mathbf{x}_i)$$

Notice that all operators have been replaced by eigenvalues.

Now reassemble the full, finite transition amplitude:

$$\begin{aligned}
\langle \mathbf{x}_N, t_N \, | \mathbf{x}_0, t_0 \rangle &= \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)}} \prod_{i=1}^{N-1} d^3 x_i d^3 p_i \left(\exp \frac{i}{\hbar} L(\mathbf{p}_i, \mathbf{x}_i) \, dt \right) \\
&= \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)}} \prod_{i=1}^{N-1} d^3 x_i d^3 p_i \left(\exp \frac{i}{\hbar} \sum_{i=1}^{N-1} L(\mathbf{p}_i, \mathbf{x}_i) \, dt \right)
\end{aligned}$$

and replacing the sum of infinitesimals by an integral,

$$\begin{aligned}
\lim_{N \to \infty, \Delta t \to 0} \exp \frac{i}{\hbar} \sum_{i=1}^{N-1} L(\mathbf{p}_i, \mathbf{x}_i) \, \Delta t &= \exp \frac{i}{\hbar} \int_{t_0}^{t_N} L(\mathbf{p}, \mathbf{x}) \, dt \\
&= \exp \frac{i}{\hbar} S[\mathbf{x}(t), \mathbf{p}(t)]
\end{aligned}$$

where $S[\mathbf{x}(t), \mathbf{p}(t)]$ is the action functional in terms of both position and momentum. Notice what has happened here. The indexed position and momentum are the position and momentum at time t_i,

$$\begin{aligned}
\mathbf{x}_i &= \mathbf{x}(t_i) \\
\mathbf{p}_i &= \mathbf{p}(t_i)
\end{aligned}$$

The remaining integrals over $\mathbf{x}_i$ and $\mathbf{p}_i$ are therefore integrals over differing $\mathbf{x}(t_i)$ and $\mathbf{p}(t_i)$, so each increment of each integration refers to a slightly different path in phase space. Since we integrate over all positions and momenta, these multiple integrals are summing the phases $\exp \frac{i}{\hbar} S[\mathbf{x}(t), \mathbf{p}(t)]$ over *every* path in phase space between the initial and final states.

With this observation in mind, we define the functional integral to be the sum over all intervening paths, in both configuration and momentum spaces:

$$\begin{aligned}
\int \mathcal{D}[\mathbf{x}(t)] &\equiv \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)/2}} \prod_{i=1}^{N-1} d^3 x_i \\
\int \mathcal{D}[\mathbf{p}(t)] &\equiv \int \cdots \int \frac{1}{(2\pi\hbar)^{3(N-1)/2}} \prod_{i=1}^{N-1} d^3 p_i
\end{aligned}$$

With this notation, the transition amplitude, or propagator, is given by

$$\langle \mathbf{x}_N, t_N | \, \mathbf{x}_0, t_0 \rangle = \int \mathcal{D}[\mathbf{x}(t)] \int \mathcal{D}[\mathbf{p}(t)] \exp \frac{i}{\hbar} S[\mathbf{x}(t), \mathbf{p}(t)]$$

This is a *phase space path integral*. Notice again that the action here is written as an independent functional of position and momentum.

The infinite products of intermediate integrals may be interpreted as meaning that the phase $\exp \frac{i}{\hbar} S[\mathbf{x}(t)]$ is to be summed over every value of position and momentum. As we shall see from examples, the result involves some curious normalizations, but the formulation is very powerful because it may be immediately generalized to field theory. Any theory of fields Φ having an action functional may be quantized by averaging $\exp \frac{i}{\hbar} S[\Phi]$ over all field configurations.

$$\begin{aligned} \langle \Phi(\mathbf{x}, t_f) | \Phi(\mathbf{x}, t_i) \rangle &= \int \mathcal{D}[\Phi(\mathbf{x}, t)] \mathcal{D}[\Pi(\mathbf{x}, t)] \exp \frac{i}{\hbar} S[\Phi(\mathbf{x}, t), \Pi(\mathbf{x}, t)] \\ S[\Phi(\mathbf{x}, t)] &= \int\limits_{t_i}^{t_f} \mathcal{L}(\Phi(\mathbf{x}, t), \Pi(\mathbf{x}, t)) d^4x \end{aligned}$$

Here, the position and time are simply parameters, while the field and its conjugate mometum are the dynamical variables.

The most important advantage of the path integral formulation is that it allows for a systematic perturbation theory. If we write the particle Lagrangian as

$$L = L_0 + V$$

and expand the exponential

$$\langle \mathbf{x}_N, t_N | \mathbf{x}_0, t_0 \rangle = \int \mathcal{D}[\mathbf{x}(t)] \int \mathcal{D}[\mathbf{p}(t)] \left(\exp \frac{i}{\hbar} \int\limits_{t_0}^{t_N} L_0 dt \right) \left(1 + \frac{i}{\hbar} \int\limits_{t_0}^{t_N} V dt + \cdots \right)$$

it is possible to evaluate the potential terms order by order. The same expansion applies to field theory,

$$\langle \Phi(\mathbf{x}, t_f) | \Phi(\mathbf{x}, t_i) \rangle = \int \mathcal{D}[\Phi(\mathbf{x}, t)] \left(\exp \frac{i}{\hbar} S_0[\Phi(\mathbf{x}, t)] \right) \left(1 + \frac{i}{\hbar} \int\limits_{t_0}^{t_N} V(\Phi) dt + \cdots \right)$$

allowing term by term approximation. Ultimately, each term in the expansion involves a different power of the potential. We keep track of the large number of required integrals by sets of *Feynman diagrams*, each diagram corresponding to a particular set of integrals.

Typically, equivalence to other methods holds, but is not demanded. The path integral is an independent hypothesis for quantization. In their book, Feynman and Hibbs show that the usual form of quantum mechanics may be derived from the Feynman path integral.

1.5 The Feynman path integral

For the form of Hamiltonian we have chosen, $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$, and for many field theories, it is possible to do all of the momentum integrals. The resulting *Feynman path integral* is a sum over curves in spacetime rather than phase space.

If we restore the finite sum expression for the action, each momentum integral is simply a Gaussian:

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}L(\mathbf{p}_i,\mathbf{x}_i)\,dt &= \frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}\left(\mathbf{p}_i\cdot\frac{(\mathbf{x}_{i+1}-\mathbf{x}_i)}{\Delta t}-\frac{\mathbf{p}_i^2}{2m}-V(\mathbf{x}_i)\right)dt \\
&= \frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}\left(-\frac{\mathbf{p}_i^2}{2m}+\mathbf{p}_i\cdot\mathbf{v}_i-V(\mathbf{x}_i)\right)dt \\
&= \frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}\left(-\frac{1}{2m}\left(\mathbf{p}_i^2-2m\mathbf{p}_i\cdot\mathbf{v}_i\right)-V(\mathbf{x}_i)\right)dt \\
&= \frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}\left(-\frac{1}{2m}\left[(\mathbf{p}_i-m\mathbf{v}_i)^2-m^2\mathbf{v}_i^2\right]-V(\mathbf{x}_i)\right)dt \\
&= \frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}\left(-\frac{1}{2m}(\mathbf{p}_i-m\mathbf{v}_i)^2+\frac{1}{2m}m^2\mathbf{v}_i^2-V(\mathbf{x}_i)\right)dt \\
&= \frac{1}{(2\pi\hbar)^3}\exp\frac{i}{\hbar}\left(\frac{1}{2}m\mathbf{v}_i^2-V(\mathbf{x}_i)\right)dt\int d^3p_i \exp\left(-\frac{i}{2m\hbar}(\mathbf{p}_i-m\mathbf{v}_i)^2\right)
\end{aligned}$$

Letting $\mathbf{y}=\mathbf{p}_i-m(\mathbf{x}_{i+1}-\mathbf{x}_i)$, the momentum integral becomes

$$\int d^3y \exp\left(-\frac{i}{2m\hbar}\mathbf{y}^2\right)$$

The imaginary unit does not really cause any problem. Adding an infinitesimal part for convergence we have

$$\int d^3y \exp\left(-\frac{i}{2m\hbar}(1-i\varepsilon)\mathbf{y}^2\right)=\int d^3y \exp\left(-\frac{\varepsilon+i}{2m\hbar}\mathbf{y}^2\right)$$

Each of the three Gaussians gives

$$\int dy \exp\left(-\alpha y^2\right) = \sqrt{\frac{\pi}{\alpha}}$$

so

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int d^3y \exp\left(-\frac{\varepsilon+i}{2m\hbar}\mathbf{y}^2\right) &= \lim_{\varepsilon\to 0}\left(\frac{2m\hbar\pi}{\varepsilon+i}\right)^{3/2} \\
&= (-2\pi i m\hbar)^{3/2}
\end{aligned}$$

The full i^{th} integral is therefore,

$$\frac{1}{(2\pi\hbar)^3}\int d^3p_i \exp\frac{i}{\hbar}L(\mathbf{p}_i,\mathbf{x}_i)\,dt = \left(\frac{m}{2\pi i\hbar}\right)^{3/2}\exp\frac{i}{\hbar}\left(\frac{1}{2}m\mathbf{v}_i^2-V(\mathbf{x}_i)\right)dt$$

Combining these back into the full path integral, we have

$$\begin{aligned}
\langle \mathbf{x}_N,t_N\,|\mathbf{x}_0,t_0\rangle &= \int\cdots\int \frac{1}{(2\pi\hbar)^{3(N-1)}}\prod_{i=1}^{N-1}d^3x_i d^3p_i\left(\exp\frac{i}{\hbar}\sum_{i=1}^{N-1}L(\mathbf{p}_i,\mathbf{x}_i)\,dt\right) \\
&= \int\cdots\int\prod_{i=1}^{N-1}d^3x_i\left(-\frac{im}{2\pi\hbar}\right)^{3(N-1)/2}\exp\frac{i}{\hbar}\int_{t_0}^{t_N}\left(\frac{1}{2}m\mathbf{v}_i^2-V(\mathbf{x}_i)\right)dt
\end{aligned}$$

and replacing the sum of infinitesimals by an integral, and defining the functional integral measure to be

$$\int\mathcal{D}[\mathbf{x}(t)]\equiv\int\cdots\int\prod_{i=1}^{N-1}d^3x_i\left(\frac{m}{2\pi i\hbar}\right)^{3N/2}$$

the transition amplitude is

$$\begin{aligned}
\langle \mathbf{x}_N, t_N \; | \mathbf{x}_0, t_0 \rangle &= \int \mathcal{D}\left[\mathbf{x}\left(t\right)\right] \exp \frac{i}{\hbar} \int\limits_{t_0}^{t_N} L\left(\mathbf{x}, \dot{\mathbf{x}}\right) dt \\
&= \int \mathcal{D}\left[\mathbf{x}\left(t\right)\right] \exp \frac{i}{\hbar} S\left[\mathbf{x}\left(t\right)\right]
\end{aligned}$$

where $S\left[\mathbf{x}\left(t\right)\right]$ is now the usual configuration space action. This is the usual form of the Feynman path integral.

The corresponding field theory *sum over histories* is

$$\langle \Phi\left(\mathbf{x}, t_f\right) | \; \Phi\left(\mathbf{x}, t_i\right)\rangle = \int \mathcal{D}\left[\Phi\left(\mathbf{x}, t\right)\right] \exp \frac{i}{\hbar} S\left[\Phi\left(\mathbf{x}, t\right)\right]$$

with the action treated as a functional of the field and its derivatives.

Chapter 2

Transition amplitudes

The initial states of scattering experiments are essentially free field. As these propagate toward one another they begin to interact, often only in a comparatively small region. Ultimately, the products of this interaction are again some other set of free fields. Recognizing this, we write the Lagrange density and corresponding Hamiltonian as a free part plus an interaction part. The two asymptotic, free states differ due to the effects of the interaction part of the Hamiltonian, which typically has a perturbative effect,

$$H = H_0 + H_{INT}$$

For example, in our $U(1)$ gauge theory coupled to the Dirac equation, we have the action

$$\begin{aligned} S &= \int \bar{\psi}\left(i\not{D} - m\right)\psi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &= \int \bar{\psi}\left(i\not{\partial} - m\right)\psi - \bar{\psi}\not{A}\psi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \end{aligned}$$

Here we see the free Dirac Lagrange density and the free electromagnetic Lagrange density separated from an interacting part:

$$\begin{aligned} \mathcal{L}_{0D} &= \bar{\psi}\left(i\not{\partial} - m\right)\psi \\ \mathcal{L}_{0EM} &= \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ \mathcal{L}_{INT} &= -\bar{\psi}\not{A}\psi \end{aligned}$$

2.1 The S matrix

We define the *scattering matrix*, or simply, the S matrix as the operator that evolves an asymptotic inital state to its time evolution as a final free state. Let the initial state be characterized in terms a complete set of eigenstates of some convenient observable,$|a(t_i)\rangle$ and the final state by another, possibly different set, $|b(t_f)\rangle$. The S-matrix is the limit as $t_f - t_i \to \infty$ of the time evolution,

$$e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)}|a(t_i)\rangle = |a(t_f)\rangle$$

so that the transition amplitude for $|a(t_i)\rangle$ evolving to $|b(t_f)\rangle$ is given by

$$\langle b(t_f)\, S\, |a(t_i)\rangle = \lim_{t_i\to-\infty, t_f\to\infty} \langle b(t_f)\, e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)}|a(t_i)\rangle$$

In practice the infinite limit simply refers to times remote enough that the states may regarded as describing free particles, far from the center of interaction.

Since the evolved final states are normalized,

$$\begin{aligned} 1 &= \langle a(t_f) | a(t_f) \rangle \\ &= \langle a(t_i) | \hat{S}^\dagger \hat{S} | a(t_i) \rangle \end{aligned}$$

This must hold for every initial state. Since these comprise a complete set of states, $\hat{S}^\dagger \hat{S}$ must be the identity

$$\hat{S}^\dagger \hat{S} = 1$$

so the scattering matrix is unitary.

Now expand

$$S = 1 + iT$$

It is convenient to compute T, since the remaining identity part of S represents noninteracting particles. Imposing unitarity,

$$\begin{aligned} 1 &= \hat{S}^\dagger \hat{S} \\ &= \left(1 - i\hat{T}^\dagger\right)\left(1 + i\hat{T}\right) \\ &= 1 + i\left(\hat{T} - \hat{T}^\dagger\right) + \hat{T}^\dagger \hat{T} \end{aligned}$$

and therefore

$$\hat{T}^\dagger \hat{T} = -i\left(\hat{T} - \hat{T}^\dagger\right)$$

In terms of matrix components, with $\hat{T}_{ba} = \langle b | \hat{T} | a \rangle$,

$$\sum_c \hat{T}^*_{ac} \hat{T}_{cb} = -i\left(\hat{T}_{ab} - \hat{T}^*_{ba}\right)$$

Looking at each diagonal element by setting $b = a$,

$$Im \hat{T}_{aa} = \frac{1}{2} \sum_c \hat{T}^*_{ac} \hat{T}_{ca}$$

thereby equating the imaginary part of each diagonal component to half the trace of the complex norm to half the corresponding component of the complex modulus.

2.2 Reducing the initial state

Recall that in field theory the dynamical variables are the fields and their conjugate momenta, so it is these that become operators, $\hat{\phi}(\mathbf{x}, t)$. This immediately suggests the Heisenberg representation, where the operators are time-dependent. Furthermore, since the space of states is found by acting with raising operators, and the raising operators are built from the fields, we may express all states in terms of fields acting on the vacuum. We make this explicit for a scalar field.

We have seen that to put the time-dependent Schrödinger states into the Heisenberg representation, we write

$$|a, t_i\rangle_H = e^{\frac{i}{\hbar}\hat{H} t_i} |a, t_i\rangle_S$$

where $|a\rangle_S = |a, t_i\rangle_S$ is an eigenket of some Schrödinger operator at time t_i. Since the Schrödinger state may evolve in time, these need not be eigenstates at other times. Elements of the S matrix are then

$$\langle b | S | a \rangle = \langle b, t_f \, | a, t_i \rangle_H$$

Recall from our study of scalar fields that the raising and lowering operators are given in terms of the scalar field operator[1]

$$\begin{aligned}
\hat{\varphi}(\mathbf{x},t) &= \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega}}\left(\hat{a}(\mathbf{k})\,e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}+\hat{a}^*(\mathbf{k})\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right)\\
\hat{\pi}(\mathbf{x},t) &= \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{\sqrt{2\omega}}\left(-i\omega\hat{a}(\mathbf{k})\,e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}+i\omega\hat{a}^*(\mathbf{k})\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\right)
\end{aligned}$$

so that inverting the Fourier transforms,

$$\begin{aligned}
\frac{1}{(2\pi)^{3/2}}\int d^3x\,\hat{\varphi}(\mathbf{x},t)\;e^{-i\mathbf{k}'\cdot\mathbf{x}} &= \frac{1}{(2\pi)^3}\int d^3x\int\frac{d^3k}{\sqrt{2\omega}}\left(\hat{a}(\mathbf{k})\,e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}e^{-i\mathbf{k}'\cdot\mathbf{x}}+\hat{a}^*(\mathbf{k})\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}e^{-i\mathbf{k}'\cdot\mathbf{x}}\right)\\
&= \int\frac{d^3k}{\sqrt{2\omega}}\left(\hat{a}(\mathbf{k})\,\delta^3(\mathbf{k}-\mathbf{k}')\,e^{-i\omega t}+\hat{a}^\dagger(\mathbf{k})\,\delta^3(\mathbf{k}+\mathbf{k}')\,e^{i\omega t}\right)\\
&= \frac{1}{\sqrt{2\omega'}}\left(\hat{a}(\mathbf{k}')\,e^{-i\omega' t}+\hat{a}^\dagger(-\mathbf{k}')\,e^{i\omega' t}\right)\\
\frac{1}{(2\pi)^{3/2}}\int d^3x\,\hat{\pi}(\mathbf{x},t)\;e^{-i\mathbf{k}'\cdot\mathbf{x}} &= \frac{1}{(2\pi)^3}\int d^3x\int\frac{d^3k}{\sqrt{2\omega}}\left(-i\omega\hat{a}(\mathbf{k})\,e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}e^{-i\mathbf{k}'\cdot\mathbf{x}}+i\omega\hat{a}^\dagger(\mathbf{k})\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}e^{-i\mathbf{k}'\cdot\mathbf{x}}\right)\\
&= \int\frac{d^3k}{\sqrt{2\omega}}\left(-i\omega\hat{a}(\mathbf{k})\,\delta^3(\mathbf{k}-\mathbf{k}')\,e^{-i\omega t}+i\omega\hat{a}^\dagger(\mathbf{k})\,\delta^3(\mathbf{k}+\mathbf{k}')\,e^{i\omega t}\right)\\
&= \frac{1}{\sqrt{2\omega'}}\left(-i\omega'\hat{a}(\mathbf{k}')\,e^{-i\omega' t}+i\omega'\hat{a}^\dagger(-\mathbf{k}')\,e^{i\omega' t}\right)
\end{aligned}$$

so combining to eliminate the conjugate terms

$$\begin{aligned}
\frac{1}{(2\pi)^{3/2}}\int d^3x\,\hat{\varphi}(\mathbf{x},t)\;e^{-i\mathbf{k}'\cdot\mathbf{x}}-\frac{1}{i\omega'}\frac{1}{(2\pi)^{3/2}}\int d^3x\,\hat{\pi}(\mathbf{x},t)\;e^{-i\mathbf{k}'\cdot\mathbf{x}} &= \frac{1}{\sqrt{2\omega'}}\left(\hat{a}(\mathbf{k}')\,e^{-i\omega' t}+\hat{a}^\dagger(-\mathbf{k}')\,e^{i\omega' t}\right)-\frac{1}{i\omega'}\frac{1}{\sqrt{2\omega'}}\Big(-i\omega\\
&= \frac{2}{\sqrt{2\omega'}}\hat{a}(\mathbf{k}')\,e^{-i\omega' t}
\end{aligned}$$

so that, dropping primes,

$$\hat{a}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\omega}{2}}\int d^3x\,\left(\hat{\varphi}(\mathbf{x},t)+\frac{i}{\omega}\hat{\pi}(\mathbf{x},t)\right)\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}$$

Taking the adjoint,

$$\hat{a}^\dagger(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\omega}{2}}\int d^3x\,\left(\hat{\varphi}(\mathbf{x},t)-\frac{i}{\omega}\hat{\pi}(\mathbf{x},t)\right)\,e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}$$

We define the convenient notation

$$f\overleftrightarrow{\partial}_\mu g\equiv f(\partial_\mu g)-(\partial_\mu f)\,g$$

This lets us write

$$\begin{aligned}
\left(\hat{\varphi}(\mathbf{x},0)+\frac{i}{\omega}\hat{\pi}(\mathbf{x},0)\right)e^{i\omega t} &= \frac{i}{\omega}\left(-i\omega e^{i\omega t}\hat{\varphi}+e^{i\omega t}\hat{\pi}\right)\\
&= \frac{i}{\omega}\left(-\partial_0 e^{i\omega t}\hat{\varphi}+e^{i\omega t}\partial_0\hat{\varphi}\right)\\
&= \frac{i}{\omega}e^{i\omega t}\overleftrightarrow{\partial_0}\hat{\varphi}
\end{aligned}$$

[1]Here we use the opposite phase convention from Part I.

and we may write the anihilation and creation operators as

$$\begin{aligned}\sqrt{2\omega}\hat{a}(\mathbf{k}) &= \frac{i}{(2\pi)^{3/2}}\int d^3x\, e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}\overleftrightarrow{\partial_0}\hat{\varphi} \\ \sqrt{2\omega}\hat{a}^\dagger(\mathbf{k}) &= -\frac{i}{(2\pi)^{3/2}}\int d^3x\, e^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}\overleftrightarrow{\partial_0}\hat{\varphi}\end{aligned}$$

Now, we define "in" and "out" fields as free fields, with an additional normalization constant, $\hat{\varphi}_{free} \to \sqrt{Z}\hat{\varphi}_{in}, \sqrt{Z}\hat{\varphi}_{out}$. The added normalization will let us take into account the scattered part of the state more easily. Now consider a transition amplitude between an initial state with m particles and a final state with n particles. We want to rewrite this as a product of field operators between vacuum states, inserting creation and anihilation operators. Inserting one,

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\,|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle &= \sqrt{2\omega}\,\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,\hat{a}^\dagger_{T_i}(\mathbf{k}_1)\,|\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle \\ &= -\frac{i}{\sqrt{Z}}\lim_{T_i\to-\infty}\int d^3x\, e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\,\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,\overleftrightarrow{\partial_0}\hat{\varphi}\,|\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle\end{aligned} \tag{2.1}$$

and we iterate this until both the bra and ket are vacuum states. Here we insert the normalization $1/\sqrt{Z}$. Note that in the Heisenberg representation, we label the operators with the time at which they are defined. For the in and out states, we will want the limit as $T_i \to -\infty$ and $T_f \to +\infty$ so that the free-field expressions are applicable.

To carry this further we need a trick or two. First, we make the problem time symmetric by introducing a vanishing term

$$\sqrt{2\omega}\,\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\left(\hat{a}^\dagger_{T_f}(\mathbf{k}_1)\,|\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle\right) = \sqrt{2\omega}\left(\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,\hat{a}_{T_f}(\mathbf{k}_1)\right)|\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle$$

where we assume that all of the final momenta $\mathbf{p}_i$ are distinct from all of the $\mathbf{k}_j$. This just means that no particle passes through the interaction without interacting. When we let $\hat{a}_{T_f}(\mathbf{k}_1)$ act to the left instead of its adjoint $\hat{a}^\dagger_{T_f}(\mathbf{k}_1)$ acting to the right, we get zero because there is no momentum $\mathbf{k}_1$ in the final state to anihilate. We may therefore write

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\,|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle &= -\sqrt{2\omega}\,\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\left(\hat{a}^\dagger_{T_f}(\mathbf{k}_1)-\hat{a}^\dagger_{T_i}(\mathbf{k}_1)\right)|\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle \\ &= \frac{i}{\sqrt{Z}}\left(\lim_{T_f\to\infty}-\lim_{T_i\to-\infty}\right)\int d^3x\, e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\,\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,\overleftrightarrow{\partial_0}\hat{\varphi}\,|\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle\end{aligned}$$

Next, we write the spatial integral as a spacetime integral of a time derivative:

$$\left(\lim_{T_f\to\infty}-\lim_{T_i\to-\infty}\right)\int d^3x f(\mathbf{x},t) = \int_{-\infty}^{\infty} dt\int d^3x\frac{\partial}{\partial t}f(\mathbf{x},t)$$

Setting

$$f(\mathbf{x},t) = \frac{i}{\sqrt{Z}}e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\overleftrightarrow{\partial_0}\hat{\varphi}$$

and substituting,

$$\left(\lim_{T_f\to\infty}-\lim_{T_i\to-\infty}\right)\int d^3x f(\mathbf{x},t) = \frac{i}{\sqrt{Z}}\int d^4x\,\partial_0\left[e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\overleftrightarrow{\partial_0}\hat{\varphi}\right] \tag{2.2}$$

Examining the integrand,

$$\begin{aligned}\partial_0\left[e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\overleftrightarrow{\partial_0}\hat{\varphi}\right] &= \partial_0\left[e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0\hat{\varphi}+i\omega_1 e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\hat{\varphi}\right] \\ &= -i\omega_1 e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0\hat{\varphi}+e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0^2\hat{\varphi}+\omega_1^2 e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\hat{\varphi}+i\omega_1 e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0\hat{\varphi} \\ &= e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\left(\partial_0^2\hat{\varphi}+\omega_1^2\hat{\varphi}\right)\end{aligned}$$

Now we use the wave equation,

$$\begin{aligned}\Box e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})} &= -m^2\varphi \\ \partial_0^2 e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})} - \nabla^2 e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})} &= -m^2\varphi \\ \left(-\omega_1^2-\nabla^2\right) e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})} &= -m^2\varphi\end{aligned}$$

so we may replace

$$e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\omega_1^2\hat{\varphi} = \left[\left(m^2-\nabla^2\right)e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\right]\hat{\varphi}$$

Returning to Eq.(2.2), substituting, and integrating the Laplacian by parts

$$\begin{aligned}\left(\lim_{T_f\to\infty}-\lim_{T_i\to-\infty}\right)\int d^3x f(\mathbf{x},t) &= \frac{i}{\sqrt{Z}}\int d^4x\partial_0\left[e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\overleftrightarrow{\partial_0}\hat{\varphi}\right] \\ &= \frac{i}{\sqrt{Z}}\int d^4x\left(e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0^2\hat{\varphi}+e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\omega_1^2\hat{\varphi}\right) \\ &= \frac{i}{\sqrt{Z}}\int d^4x\left(e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0^2\hat{\varphi}+\left[\left(m^2-\nabla^2\right)e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\right]\hat{\varphi}\right) \\ &= \frac{i}{\sqrt{Z}}\int d^4x\left(e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\partial_0^2\hat{\varphi}+e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\left(m^2-\nabla^2\right)\hat{\varphi}\right) \\ &= \frac{i}{\sqrt{Z}}\int d^4x e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\left(\partial_0^2\hat{\varphi}-\nabla^2+m^2\hat{\varphi}\right) \\ &= \frac{i}{\sqrt{Z}}\int d^4x e^{-ik_{1\alpha}x^\alpha}\left(\Box+m^2\right)\hat{\varphi}\end{aligned}$$

so that finally,

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\left|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\right\rangle &= \frac{i}{\sqrt{Z}}\left(\lim_{T_f\to\infty}-\lim_{T_i\to-\infty}\right)\int d^3x\, e^{-i(\omega_1 t-\mathbf{k}_1\cdot\mathbf{x})}\left\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right|\overleftrightarrow{\partial_0}\hat{\varphi}\left|\mathbf{k}_2,\ldots,\mathbf{k}_m\right\rangle \\ &= \frac{i}{\sqrt{Z}}\int d^4x\, e^{-ik_{1\alpha}x^\alpha}\left(\Box+m^2\right)\left\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right|\hat{\varphi}\left|\mathbf{k}_2,\ldots,\mathbf{k}_m\right\rangle \qquad (2.3)\end{aligned}$$

and we see that the result is completely covariant, and expressed in terms of an expectation value of the field. By iterating this procedure, we may express the transition amplitude in terms of *vacuum* expectation values of the field.

2.2.1 Time ordering

To handle the final states, we first need to understand time ordering.

Suppose we wish to solve the Schrödinger equation when the Hamiltonian operator depends on time. Then we want to solve

$$\hat{H}(t)\left|\psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle$$

For a small time step, $\Delta t = t - t_0$, we have to first order

$$\left|\psi(t)\right\rangle = \left|\psi(t_0)\right\rangle - \frac{i}{\hbar}\int_{t_0}^{t_1}\hat{H}(t)\left|\psi(t)\right\rangle dt$$

since then

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left|\psi\left(t\right)\right\rangle &= i\hbar\frac{\partial}{\partial t}\left(\left|\psi\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t_1}\hat{H}\left(t\right)\left|\psi\left(t\right)\right\rangle dt\right) \\
&= \frac{\partial}{\partial t}\int\limits_{t_0}^{t}\hat{H}\left(t\right)\left|\psi\left(t\right)\right\rangle dt \\
&= \hat{H}\left(t\right)\left|\psi\left(t\right)\right\rangle
\end{aligned}$$

This is the first Born approximation, and we may iterate it,

$$\begin{aligned}
\left|\psi\left(t_1\right)\right\rangle &= \left|\psi\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t_1}\hat{H}\left(t\right)\left|\psi\left(t\right)\right\rangle dt \\
&= \left|\psi\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t_1}\hat{H}\left(t\right)\left(\left|\psi\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'\right) dt \\
&= \left|\psi\left(t_0\right)\right\rangle - \frac{i}{\hbar}\int\limits_{t_0}^{t_1}\hat{H}\left(t\right)\left|\psi\left(t_0\right)\right\rangle + \left(\frac{i}{\hbar}\right)^2\int\limits_{t_0}^{t_1}\hat{H}\left(t\right)\int\limits_{t_0}^{t}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt
\end{aligned}$$

The process may be continued to arbitrary order.

To make the result look like an exponential, we consider the double integral:

$$I_1 \equiv \int\limits_{t_1}^{t}\hat{H}\left(t\right)\int\limits_{t_0}^{t}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt$$

Notice that the first integral runs from t_0 to t_1 while the second starts at t_1 and extends to the final time t. Therefore, the operator product $\hat{H}\left(t\right)\hat{H}\left(t'\right)$ is *time ordered*, in the sense that $t > t'$ throughout the range of integration. Our expression would take a simpler form if we could extend both integrals over the full range from t_0 to t. The t' integral already has this range, but the t integral is missing

$$I_2 \equiv \int\limits_{t_0}^{t_1}\hat{H}\left(t\right)\int\limits_{t}^{t_1}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt$$

However, in this integral, $t < t'$ throughout.

We define the *time ordered product*, T, to be

$$T\left(\hat{H}\left(t\right)\hat{H}\left(t'\right)\right) \equiv \left\{\begin{array}{ll} \hat{H}\left(t\right)\hat{H}\left(t'\right) & t > t' \\ \hat{H}\left(t'\right)\hat{H}\left(t\right) & t' > t \end{array}\right.$$

Using this, the two integrals become equal,

$$T\left(I_1\right) = T\left(I_2\right)$$

and therefore,

$$\begin{aligned}
I_1 &\equiv T\left(I_1\right) \\
&= \frac{1}{2}\left(T\left(I_1\right)+T\left(I_2\right)\right) \\
&= \frac{1}{2}T\left(I_1+I_2\right) \\
&= \frac{1}{2}\int_{t_0}^{t_1}\hat{H}\left(t\right)\int_{t_0}^{t}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt \\
&= \frac{1}{2}T\left(\int_{t_0}^{t_1}\hat{H}\left(t\right)\int_{t_0}^{t}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt+\int_{t_0}^{t_1}\hat{H}\left(t\right)\int_{t}^{t_1}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt\right) \\
&= \frac{1}{2}T\left(\int_{t_0}^{t_1}\hat{H}\left(t\right)\int_{t_0}^{t_1}\hat{H}\left(t'\right)\left|\psi\left(t'\right)\right\rangle dt'dt\right)
\end{aligned}$$

Now both integrals have the full range. If we continue this process for higher terms in the Born approximation, the cubic term requires a factor of $\frac{1}{3!}$, the fourther $\frac{1}{4!}$ and so on. At each order the term has the form of the corresponding term in the power series for an exponential, so formally, we write the full series as the *time ordered exponential*, $Te^{-\frac{i}{\hbar}\int_{t_0}^{t_1}\hat{H}(t)dt}$ so that

$$\left|\psi\left(t\right)\right\rangle = Te^{-\frac{i}{\hbar}\int_{t_0}^{t_1}\hat{H}(t)dt}\left|\psi\left(t_0\right)\right\rangle$$

Always remember that the exponential of an operator is always just compact notation for the original power series.

2.3 Reducing the final state

By iterating the procedure for rewriting the initial state as in Eq.(2.3), we may write the transition amplitude, Eq.(2.1), as

$$\left\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\right|\left.\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\right\rangle = \left(\frac{i}{\sqrt{Z}}\right)^m\left(\prod_{s=1}^{m}\int d^4x_{(s)}\,e^{-ik_{(s)\alpha}x^{\alpha}_{(s)}}\left(\Box_s+m^2\right)\right)\left\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right|\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots$$

but the spacetime coordinates are integrated over all times, and we must require the field operators to be time ordered,

$$\left\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\right|\left.\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\right\rangle = \left(\frac{i}{\sqrt{Z}}\right)^m\left(\prod_{s=1}^{m}\int d^4x_{(s)}\,e^{-ik_{(s)\alpha}x^{\alpha}_{(s)}}\left(\Box_s+m^2\right)\right)\left\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right|T\left(\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots\right.$$

There is no combinatoric factor required, since all of the integrals already have the full range of the time coordinate.

To accomplish the same reduction of the outgoing states, we write

$$\left\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right| = \sqrt{2\omega}\left\langle \mathbf{p}_2,\ldots,\mathbf{p}_n\right|\hat{a}_{T_f}\left(\mathbf{p}_1\right)$$

As for the incoming states, we may make the expression time symmetric by replacing $\hat{a}_{T_f}\left(\mathbf{p}_1\right)$ by the difference, $\hat{a}_{T_f}\left(\mathbf{p}_1\right)-\hat{a}_{T_i}\left(\mathbf{p}_1\right)$. This still works because of the time ordering; starting with

$$\frac{i}{\sqrt{Z}}\left(\prod_{s=1}^{m}\int d^4x_{(s)}\,e^{-ik_{(s)\alpha}x^{\alpha}_{(s)}}\left(\Box_s+m^2\right)\right)\sqrt{2\omega}\left\langle \mathbf{p}_2,\ldots,\mathbf{p}_n\right|T\left(\hat{a}_{T_i}\left(\mathbf{p}_1\right)\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots\hat{\varphi}\left(x_m\right)\right)\left|0\right\rangle$$

we see that

$$T\left(\hat{a}_{T_i}\left(\mathbf{p}_1\right)\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots\hat{\varphi}\left(x_m\right)\right)=T\left(\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots\hat{\varphi}\left(x_m\right)\hat{a}_{T_i}\left(\mathbf{p}_1\right)\right)$$

since the initial time of $\hat{a}_{T_i}\left(\mathbf{p}_1\right)$ is always earliest and therefore rightmost. Then, when $\hat{a}_{T_i}\left(\mathbf{p}_1\right)$ acts on the vacuum, it gives zero.

Now we may use the same integration replacement as before, the only difference being that we take the adjoint. This only alters the phase to $e^{+ip_{1\alpha}y^{\alpha}}$ and we have

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\left|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\right\rangle &= \left(\frac{i}{\sqrt{Z}}\right)^{n+m}\left(\prod_{r=1}^{n}\int d^4y_{(r)}\,e^{+ip_{(r)\alpha}y^{\alpha}_{(r)}}\left(\Box_r+m^2\right)\right)\left(\prod_{s=1}^{m}\int d^4x_{(s)}\,e^{-ik_{(s)\alpha}x^{\alpha}_{(s)}}\left(\Box_s\right.\right.\\ &\quad\times\langle 0|\,T\left(\hat{\varphi}\left(y_1\right)\hat{\varphi}\left(y_2\right)\ldots\hat{\varphi}\left(y_n\right)\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots\hat{\varphi}\left(x_m\right)\right)|0\rangle\end{aligned}$$

Combining expressions,

$$\begin{aligned}\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\left|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\right\rangle &= \left(\frac{i}{\sqrt{Z}}\right)^{n+m}\prod_{r=1}^{n}\prod_{s=1}^{m}\int d^4y_{(r)}\int d^4x_{(s)}\,e^{ip_{(r)\alpha}y^{\alpha}_{(r)}-ik_{(s)\alpha}x^{\alpha}_{(s)}}\left(\Box_r+m^2\right)\left(\Box_s+m^2\right)\\ &\quad\times\langle 0|\,T\left(\hat{\varphi}\left(y_1\right)\hat{\varphi}\left(y_2\right)\ldots\hat{\varphi}\left(y_n\right)\hat{\varphi}\left(x_1\right)\hat{\varphi}\left(x_2\right)\ldots\hat{\varphi}\left(x_m\right)\right)|0\rangle\end{aligned}$$

This transition amplitude is written in terms of the Heisenberg states; for the Schrödinger states we write instead

$$\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f\left|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\right\rangle_H=\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,S\,|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle_S$$

Using the expansion of the scattering matrix,

$$S=1+iT$$

and remembering that there is no overlap in the outgoing momenta $\left(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right)$ with the incoming $\left(\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m\right)$, we see that

$$\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,1\,|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle_S = 0$$

This means that we have computed the *scattered* part of the S matrix,

$$\langle\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n|\,iT\,|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m\rangle_S$$

2.4 Green functions

Consider the expression for a single particle transition,

$$\langle\mathbf{p}_1;T_f\left|\mathbf{k}_1;T_i\right\rangle = \left(\frac{i}{\sqrt{Z}}\right)^2\int d^4y\int d^4x\,e^{ip_\alpha y^\alpha-ik_\alpha x^\alpha}\left(\Box_y+m^2\right)\left(\Box_x+m^2\right)\langle 0|\,T\left(\hat{\varphi}\left(y\right)\hat{\varphi}\left(x\right)\right)|0\rangle$$

This has the form of a Fourier transform,

$$\int d^4x\,e^{-ik_\alpha x^\alpha}\left(\Box_x+m^2\right)\hat{\varphi}\left(x\right)$$

The Green function for the free field would satisfy

$$\left(\Box_x+m^2\right)G\left(x,x'\right)=-4\pi\delta^4\left(x-x'\right)$$

Correspondingly, if the fields were replaced by Green functions, the transition amplitude would be

$$
\begin{aligned}
\langle \mathbf{p}_1; T_f \; |\mathbf{k}_1; T_i\rangle &= \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4y \int d^4x \, e^{ip_\alpha y^\alpha - ik_\alpha x^\alpha} \left(\Box_y + m^2\right)\left(\Box_x + m^2\right) \langle 0|\, T\left(G\left(y, y'\right) G\left(x, x'\right)\right) |0\rangle \\
&= \left(\frac{i}{\sqrt{Z}}\right)^2 16\pi^2 \int d^4y \int d^4x \, e^{ip_\alpha y^\alpha - ik_\alpha x^\alpha} \delta^4\left(y - y'\right) \delta^4\left(x - x'\right) \langle 0 \,|0\rangle \\
&= \left(\frac{i}{\sqrt{Z}}\right)^2 16\pi^2 e^{ip_\alpha y'^\alpha - ik_\alpha x'^\alpha} \\
&= \left(\frac{i}{\sqrt{Z}}\right)^2 16\pi^2 \left(2\pi\right)^8 \delta^4\left(p_\alpha\right) \int d^4y' \int d^4x' \, e^{ip_\alpha y'^\alpha - ik_\alpha x'^\alpha}
\end{aligned}
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