Quantization of scalar fields

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We have introduced several distinct types of fields, with actions that give their field equations. These include scalar fields,

$$S = \frac{1}{2}\int \left(\partial^\alpha\varphi\partial_\alpha\varphi - m^2\varphi^2\right) d^4x \tag{1}$$

and complex scalar fields,

$$S = \frac{1}{2}\int \left(\partial^\alpha\varphi^*\partial_\alpha\varphi - m^2\varphi^*\varphi\right) d^4x \tag{2}$$

These are often called *charged scalar fields* because they have a nontrivial global $U(1)$ symmetry that allows them to couple to electromagnetic fields. Scalar fields have spin 0 and mass m.

The next possible value of $W^2 \sim J^2$ is spin-$\frac{1}{2}$, which is possessed by spinors. Dirac spinors satisfy the Dirac equation, which follows from the action

$$S = \int d^4x\ \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{3}$$

Once again, the mass is m. For higher spin, we have the zero mass, spin-1 electromagnetic field, with action

$$S = \int d^4x\ \left(\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + J^\alpha A_\alpha\right) \tag{4}$$

Electromagnetic theory has an important generalization in the Yang-Mills field, $F^A{}_{\alpha\beta}$ where the additional index corresponds to an $SU(n)$ symmetry. We could continue with the spin-$\frac{3}{2}$ Rarita-Schwinger field and the spin-2 metric field, $g_{\alpha\beta}$ of general relativity. The latter follows the Einstein-Hilbert action,

$$S = \int d^4x\ \sqrt{-\det\left(g_{\alpha\beta}\right)}g^{\alpha\beta}R^\mu{}_{\alpha\mu\beta} \tag{5}$$

where $R^\mu{}_{\nu\alpha\beta}$ is the Riemann curvature tensor computed from $g_{\alpha\beta}$ and its first and second derivatives, while the Rarita-Schwinger is most consistently treated as the supersymmetric partner of the graviton.

We first turn our attention to scalar and charged scalar fields.

We need the Hamiltonian formulation of field theory to do this properly, and we will immediately see the need for functional differentiation.

1 Hamilton's equations for the Klein-Gordon (scalar) field

To begin quantization, we require the Hamiltonian formulation of scalar field theory. This, in turn, requires the momentum conjugate to the field φ. In mechanics, this is simply $\frac{\partial L}{\partial \dot{q}}$, but in field theory, the Lagrangian is a functional rather than a function.

To see this explicitly, rewrite the Klein-Gordon action, eq(1), as

$$S = \int \left[\frac{1}{2}\int \left(\partial^\alpha\varphi\partial_\alpha\varphi - m^2\varphi^2\right) d^3x\right] dt$$

where we identify the term in brackets as the Lagrangian,

$$L = \frac{1}{2} \int \left(\partial^{\alpha} \varphi \partial_{\alpha} \varphi - m^2 \varphi^2 \right) d^3 x$$

Therefore, to find the conjugate momentum density to φ, we must take a *functional derivative* of the Lagrangian with respect to $\dot{\varphi}$,

$$\begin{aligned} \pi &\equiv \frac{\delta L}{\delta \left(\partial_0 \varphi \right)} \\ &= \frac{\delta}{\delta \left(\partial_0 \varphi \right)} \frac{1}{2} \int \left(\partial^{\alpha} \varphi \partial_{\alpha} \varphi - m^2 \varphi^2 \right) d^3 x' \\ &= \int \left(\partial^0 \varphi \right) \delta^3 (\mathbf{x}' - \mathbf{x}) d^3 x' \\ &= \partial^0 \varphi(\mathbf{x}) \\ &= \dot{\varphi}(\mathbf{x}) \end{aligned}$$

Notice that we treat $\varphi(\mathbf{x})$ and its time derivative $\dot{\varphi}(\mathbf{x})$ as independent.

In field theory, we think of the coordinates $(\mathbf{x}, t)$ as parameters or labels in the same way that indices label the components of coordinates and momentum in mechanics. Thus, the expression for the Hamiltonian changes from the discrete sum over i, $H = \sum p_i \dot{q}^i - L$ to an integral over the continuous parameters $\mathbf{x}$

$$H = \int \pi(\mathbf{x}) \dot{\varphi}(\mathbf{x}) d^3 x - L$$

or, expanding the Lagrangian,

$$\begin{aligned} H &= \int \pi(\mathbf{x}) \dot{\varphi}(\mathbf{x}) d^3 x - \frac{1}{2} \int \left(\dot{\varphi}(\mathbf{x})^2 - \boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} \varphi - m^2 \varphi^2 \right) d^3 x \\ &= \frac{1}{2} \int \left(\pi^2 + \boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} \varphi + m^2 \varphi^2 \right) d^3 x \end{aligned}$$

We define the *Hamiltonian density,*

$$\mathcal{H} = \frac{1}{2} \left(\pi^2 + \nabla \varphi \cdot \quad \nabla \varphi + m^2 \varphi^2 \right) \tag{6}$$

Hamilton's equations are also functional derivatives. Since the Hamiltonian is now a functional, we replace the mechanical form of Hamilton's equations,

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \end{aligned}$$

with their functional derivative generalizations,

$$\begin{aligned} \dot{\varphi}(\mathbf{x}) &= \frac{\delta H}{\delta \pi(\mathbf{x})} \\ \dot{\pi}(\mathbf{x}) &= -\frac{\delta H}{\delta \varphi(\mathbf{x})} \end{aligned}$$

and check that this reproduces the correct field equation. Taking the indicated derivative for $\dot{\varphi}$ gives

$$\begin{aligned}
\dot{\varphi}(\mathbf{x}) &= \frac{\delta H}{\delta \pi(\mathbf{x})} \\
&= \frac{1}{2}\frac{\delta}{\delta \pi(\mathbf{x})}\int \left(\pi^2 + \nabla\varphi\cdot\nabla\varphi + m^2\varphi^2\right) d^3x' \\
&= \frac{1}{2}\int \left(2\pi(\mathbf{x}')\frac{\delta \pi(\mathbf{x}')}{\delta \pi_i(\mathbf{x})}\right) d^3x' \\
&= \int \pi(\mathbf{x}')\delta^3(\mathbf{x}' - \mathbf{x})d^3x' \\
&= \pi(\mathbf{x})
\end{aligned}$$

while for $\dot{\pi}$,

$$\begin{aligned}
\dot{\pi}(\mathbf{x}) &= -\frac{\delta H}{\delta \varphi(\mathbf{x})} \\
&= -\frac{1}{2}\frac{\delta}{\delta \varphi(\mathbf{x})}\int \left(\pi^2 + \nabla\varphi\cdot\nabla\varphi + m^2\varphi^2\right) d^3x' \\
&= -\int \left(\nabla\varphi\cdot\nabla\frac{\delta\varphi(\mathbf{x}')}{\delta\varphi(\mathbf{x})} + m^2\varphi\frac{\delta\varphi(\mathbf{x}')}{\delta\varphi(\mathbf{x})}\right) d^3x' \\
&= \int \left(\nabla^2\varphi\cdot\frac{\delta\varphi(\mathbf{x}')}{\delta\varphi(\mathbf{x})} - m^2\varphi\frac{\delta\varphi(\mathbf{x}')}{\delta\varphi(\mathbf{x})}\right) d^3x' \\
&= \int \left(\nabla^2\varphi\delta^3(\mathbf{x}' - \mathbf{x}) - m^2\varphi\delta^3(\mathbf{x}' - \mathbf{x})\right) d^3x' \\
&= \nabla^2\varphi - m^2\varphi
\end{aligned}$$

But substituting $\dot{\pi} = \partial_0\dot{\varphi} = \ddot{\varphi}$ for $\dot{\pi}$,

$$\begin{aligned}
\ddot{\varphi} &= \nabla^2\varphi - m^2\varphi \\
\Box\varphi &= -m^2\varphi
\end{aligned}$$

we recover the Klein-Gordon field equation.

2 Functional Poisson brackets

We move toward quantization by writing the field equations in terms of functional Poisson brackets. Here too, we expect dynamical variables f and g to be functionals so the bracket becomes

$$\{f(\varphi,\pi), g(\varphi,\pi)\} \equiv \int \left(\frac{\delta f}{\delta\pi(\mathbf{x})}\frac{\delta g}{\delta\varphi(\mathbf{x})} - \frac{\delta f}{\delta\varphi(\mathbf{x})}\frac{\delta g}{\delta\pi(\mathbf{x})}\right) d^3x$$

where we replaced the sum over all p_i and q^i by an integral over all $\mathbf{x}$. The bracket is evaluated at a constant time. Then we have

$$\begin{aligned}
\{\pi(\mathbf{x}'), \varphi(\mathbf{x}'')\} &= \int \left(\frac{\delta\pi(\mathbf{x}')}{\delta\pi(\mathbf{x})}\frac{\delta\varphi(\mathbf{x}'')}{\delta\varphi(\mathbf{x})} - \frac{\delta\pi(\mathbf{x}')}{\delta\varphi(\mathbf{x})}\frac{\delta\varphi(\mathbf{x}'')}{\delta\pi(\mathbf{x})}\right) d^3x \\
&= \int \delta^3(\mathbf{x}' - \mathbf{x})\delta^3(\mathbf{x}'' - \mathbf{x})d^3x \\
&= \delta^3(\mathbf{x}'' - \mathbf{x}')
\end{aligned}$$

while

$$\{\pi(\mathbf{x}'), \pi(\mathbf{x}'')\} = \{\varphi(\mathbf{x}'), \varphi(\mathbf{x}'')\} = 0$$

Hamilton's equations work out correctly using the functional Poisson brackets,

$$\begin{aligned}
\dot{\varphi}(\mathbf{x}) &= \{H(\varphi,\pi),\varphi(\mathbf{x}')\} \\
&= \int\left(\frac{\delta H(\varphi,\pi)}{\delta\pi(\mathbf{x})}\frac{\delta\varphi(\mathbf{x}')}{\delta\varphi(\mathbf{x})}-\frac{\delta H}{\delta\varphi(\mathbf{x})}\frac{\delta\varphi(\mathbf{x}')}{\delta\pi(\mathbf{x})}\right)d^3x \\
&= \int\frac{\delta H(\varphi,\pi)}{\delta\pi(\mathbf{x})}\delta^3(\mathbf{x}-\mathbf{x}')d^3x \\
&= \frac{\delta H(\varphi(\mathbf{x}),\pi(\mathbf{x}))}{\delta\pi(\mathbf{x})}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\pi}(\mathbf{x}) &= \{H(\varphi,\pi),\pi(\mathbf{x}')\} \\
&= \int\left(\frac{\delta H(\varphi,\pi)}{\delta\pi(\mathbf{x})}\frac{\delta\pi(\mathbf{x}')}{\delta\varphi(\mathbf{x})}-\frac{\delta H}{\delta\varphi(\mathbf{x})}\frac{\delta\pi(\mathbf{x}')}{\delta\pi(\mathbf{x})}\right)d^3x \\
&= -\int\frac{\delta H(\varphi,\pi)}{\delta\varphi(\mathbf{x})}\delta^3(\mathbf{x}-\mathbf{x}')d^3x \\
&= -\frac{\delta H(\varphi(\mathbf{x}),\pi(\mathbf{x}))}{\delta\varphi(\mathbf{x})}
\end{aligned}$$

Now we quantize, canonically. The field and its conjugate momentum become operators and the fundamental Poisson brackets become commutators,

$$\{\pi(\mathbf{x}'),\varphi(\mathbf{x}'')\}=\delta^3(\mathbf{x}''-\mathbf{x}')\Rightarrow[\hat{\pi}(\mathbf{x}'),\hat{\varphi}(\mathbf{x}'')]=i\delta^3(\mathbf{x}''-\mathbf{x}')$$

(where $\hbar=1$) while

$$[\hat{\varphi}(\mathbf{x}'),\hat{\varphi}(\mathbf{x}'')]=[\hat{\pi}(\mathbf{x}'),\hat{\pi}(\mathbf{x}'')]=0$$

These are the fundamental commutation relations of the quantum field theory. Because the commutator of the field operators $\hat{\pi}(\mathbf{x})$ and $\hat{\varphi}(\mathbf{x})$ are evaluated at the same value of t, these are called *equal time commutation relations.* More explicitly,

$$\begin{aligned}
[\hat{\pi}(\mathbf{x}',t),\hat{\varphi}(\mathbf{x}'',t)] &= i\delta^3(\mathbf{x}''-\mathbf{x}') \\
[\hat{\varphi}(\mathbf{x}',t),\hat{\varphi}(\mathbf{x}'',t)] &= [\hat{\pi}(\mathbf{x}',t),\hat{\pi}(\mathbf{x}'',t)]=0
\end{aligned}\tag{7}$$

This completes the canonical quantization. The trick, of course, is to find some solutions that have the required quantized properties.

3 Classical solution for the free Klein-Gordon field

Having written commutation relations for the field, we still have the problem of finding solutions and interpreting them. To begin, we look at solutions of the classical theory. The field equation

$$\Box\varphi=-\frac{m^2}{\hbar^2}\varphi$$

(where we have replaced $\hbar$, but retain $c=1$) is not hard to solve as a superposition of plane waves. Consider the conjugate plane waves,

$$\begin{aligned}
\varphi(\mathbf{x},t) &= Ae^{\frac{i}{\hbar}(p_\alpha x^\alpha)}+A^\dagger e^{-\frac{i}{\hbar}(p_\alpha x^\alpha)} \\
&= Ae^{\frac{i}{\hbar}(Et-\mathbf{p}\cdot\mathbf{x})}+A^\dagger e^{-\frac{i}{\hbar}(Et-\mathbf{p}\cdot\mathbf{x})}
\end{aligned}$$

where at this point, $A^\dagger$ is simply the complex conjugate, A^*. Substituting into the field equation we have

$$A\left(\frac{i}{\hbar}\right)^2 p_\alpha p^\alpha \exp\frac{i}{\hbar}\left(p_\alpha x^\alpha\right) = -\frac{m^2}{\hbar^2} A\exp\frac{i}{\hbar}\left(p_\alpha x^\alpha\right)$$

so we need the usual mass-energy-momentum relation $p_\alpha p^\alpha = m^2$. Solving for the energy, we keep both solutions,

$$\begin{aligned} E_+ &= \sqrt{\mathbf{p}^2 + m^2} \\ E_- &= -\sqrt{\mathbf{p}^2 + m^2} \end{aligned}$$

then construct the general solution by Fourier superposition. To keep the result manifestly relativistic, we use a Dirac delta function to impose $p_\alpha p^\alpha = m^2$. We also insert a unit step function, $\Theta(E)$, to insure positivity of the energy. This insertion may seem a bit *ad hoc*, and it is – we will save discussion of the negative energy solutions and antiparticles for later. Then, integrating over all energies and momenta,

$$\varphi(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int \sqrt{2E}\left(a\left(E,\mathbf{p}\right)e^{\frac{i}{\hbar}p_\alpha x^\alpha} + a^\dagger\left(E,\mathbf{p}\right)e^{-\frac{i}{\hbar}p_\alpha x^\alpha}\right)\delta\left(p_\alpha p^\alpha - m^2\right)\Theta\left(E\right)\hbar^{-4}d^4p$$

where $A = \sqrt{2E}\,a(E,\mathbf{p})$ is the arbitrary complex amplitude of each wave mode and $\frac{1}{(2\pi)^{3/2}}$ is the conventional normalization for Fourier integrals.

Recall that for a function $f(x)$ with zeros at x_i, $i = 1, 2, \ldots, n$, $\delta(f)$ gives a contribution at each zero:

$$\delta(f) = \sum_{i=1}^{n}\frac{1}{|f'(x_i)|}\delta(x - x_i) \tag{8}$$

so the quadratic delta function can be written as

$$\begin{aligned} \delta\left(p_\alpha p^\alpha - m^2\right) &= \delta\left(E^2 - \mathbf{p}^2 - m^2\right) \\ &= \frac{1}{2|E|}\delta\left(E - \sqrt{\mathbf{p}^2 + m^2}\right) + \frac{1}{2|E|}\delta\left(E + \sqrt{\mathbf{p}^2 + m^2}\right) \end{aligned}$$

Exercise: Prove eq.(8).

Exercise: Argue that $\Theta\left(E\right)$ is Lorentz invariant.

The integral for the solution $\varphi\left(\mathbf{x},t\right)$ becomes

$$\begin{aligned} \varphi\left(\mathbf{x},t\right) &= \frac{1}{(2\pi)^{3/2}}\int \sqrt{2E}\left(ae^{\frac{i}{\hbar}p_\alpha x^\alpha} + a^\dagger e^{-\frac{i}{\hbar}p_\alpha x^\alpha}\right)\frac{1}{2|E|}\delta\left(E - \sqrt{\mathbf{p}^2 + m^2}\right)\Theta(E)\hbar^{-4}d^4p \\ &\quad + \frac{1}{(2\pi)^{3/2}}\int \sqrt{2E}\left(ae^{\frac{i}{\hbar}p_\alpha x^\alpha} + a^\dagger e^{-\frac{i}{\hbar}p_\alpha x^\alpha}\right)\frac{1}{2|E|}\delta\left(E + \sqrt{\mathbf{p}^2 + m^2}\right)\Big\}\Theta(E)\hbar^{-4}d^4p \\ &= \frac{1}{(2\pi)^{3/2}\hbar^4}\int \frac{d^4p}{\sqrt{2|E|}}\left(ae^{\frac{i}{\hbar}p_\alpha x^\alpha} + a^\dagger e^{-\frac{i}{\hbar}p_\alpha x^\alpha}\right)\delta\left(E - \sqrt{\mathbf{p}^2 + m^2}\right) \end{aligned}$$

Performing the energy integral using the delta function,

$$\begin{aligned} \varphi\left(\mathbf{x},t\right) &= \frac{1}{(2\pi)^{3/2}\hbar^4}\int \frac{d^4p}{\sqrt{2|E|}}\left(a\left(p_\alpha\right)e^{\frac{i}{\hbar}p_\alpha x^\alpha} + a^\dagger\left(p_\alpha\right)e^{-\frac{i}{\hbar}p_\alpha x^\alpha}\right)\delta\left(E - \sqrt{\mathbf{p}^2 + m^2}\right) \\ &= \frac{1}{(2\pi)^{3/2}\hbar^3}\int \frac{d^3p}{\sqrt{2|E\left(\mathbf{p}\right)|}}\left(a\left(\mathbf{p}\right)e^{\frac{i}{\hbar}(E(\mathbf{p})t - \mathbf{p}\cdot\mathbf{x})} + a^\dagger\left(\mathbf{p}\right)e^{-\frac{i}{\hbar}(E(\mathbf{p})t - \mathbf{p}\cdot\mathbf{x})}\right) \end{aligned}$$

Notice that we may write $a(\mathbf{p})$ for $a(E(\mathbf{p}),\mathbf{p})$. Define the 4-vector,

$$\begin{aligned} k^\mu &= (\omega, \mathbf{k}) \\ \mathbf{k} &= \frac{\mathbf{p}}{\hbar} \\ \omega &= \frac{1}{\hbar}\sqrt{\mathbf{p}^2 + m^2} = \sqrt{\mathbf{k}^2 + \left(\frac{m}{\hbar}\right)^2} \end{aligned}$$

Then

$$\varphi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} \left(a(\mathbf{k}) e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k}) e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \right) \tag{9}$$

This is the general classical solution for the Klein-Gordon field. Again, the amplitudes a and $a^\dagger$ depend only on $\mathbf{k}$.

To check that our solution satisfies the Klein-Gordon equation, we need only apply the wave operator to the right side. This pulls down an overall factor of $(ik_\mu)(ik^\mu) = -\frac{1}{\hbar^2}\left(E^2 - \mathbf{p}^2\right) = -\frac{m^2}{\hbar^2}$. Since this is constant, it comes out of the integral, giving $-\frac{m^2}{\hbar^2}\varphi$ as required.

To quantize this form of the solution, φ becomes an operator. Within the Fourier expansion, the only way to achieve this is to allow the mode amplitudes to become operators, $\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k})$. We need to find the commutation relations satisfied by these amplitudes.

4 Quantization of the solution

Classically, the full evoltion of the field is determined by the initial field and initial momentum. These are sufficient to determine the amplitudes a and $a^\dagger$ in terms of initial conditions, and these are the relationships we require. Since we know the commutation relations that $\hat{\varphi}$ and $\hat{\pi}$ satisfy as operators, knowing the amplitudes $\hat{a}$ and $\hat{a}^\dagger$ in terms of (φ, π) lets us find their commutation relations.

4.1 Solving for the mode amplitudes

To this end, multiply $\varphi(\mathbf{x}, t)$ by $\frac{1}{(2\pi)^{3/2}} d^3x e^{i\mathbf{k}'\cdot\mathbf{x}}$ and integrate. On the left this gives the Fourier transform of the field, while the right side gives a combination of the amplitudes,

$$\begin{aligned} \frac{1}{(2\pi)^{3/2}} \int \varphi(\mathbf{x}, 0) e^{i\mathbf{k}'\cdot\mathbf{x}} d^3x &= \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2\omega}} \left(a(\mathbf{k}) e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} + a^\dagger(\mathbf{k}) e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}} \right) d^3x \\ &= \int \frac{d^3k}{\sqrt{2\omega}} \left(a(\mathbf{k}) \delta^3(\mathbf{k}' - \mathbf{k}) + a^\dagger(\mathbf{k}) \delta^3(\mathbf{k}' + \mathbf{k}) \right) \\ &= \frac{1}{\sqrt{2\omega}} \left(a(\mathbf{k}') + a^\dagger(-\mathbf{k}') \right) \end{aligned}$$

We also need to invert the expression for the conjugate momentum, given by the time derivative of π,

$$\begin{aligned} \pi(\mathbf{x}, t) &= \partial_0 \varphi(\mathbf{x}, t) \\ &= \frac{i}{(2\pi)^{3/2}} \int \sqrt{\frac{\omega}{2}} d^3k \left(a(\mathbf{k}) e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} - a^\dagger(\mathbf{k}) e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \right) \end{aligned}$$

Taking the Fourier transform of the momentum density we find

$$\frac{1}{(2\pi)^{3/2}} \int \pi(\mathbf{x}, 0) e^{i\mathbf{k}'\cdot\mathbf{x}} d^3x = \frac{i}{(2\pi)^3} \int \sqrt{\frac{\omega'}{2}} d^3k \left(a(\mathbf{k}) e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}} \right) d^3x$$

$$
\begin{aligned}
&= i\int\sqrt{\frac{\omega'}{2}}d^3k\left(a\left(\mathbf{k}\right)\delta^3\left(\mathbf{k}'-\mathbf{k}\right)-a^\dagger\left(\mathbf{k}\right)\delta^3\left(\mathbf{k}'+\mathbf{k}\right)\right)\\
&= i\sqrt{\frac{\omega'}{2}}\left(a\left(\mathbf{k}'\right)-a^\dagger\left(-\mathbf{k}'\right)\right)
\end{aligned}
$$

Now we solve for the amplitudes,

$$
\begin{aligned}
a\left(\mathbf{k}'\right)+a^\dagger\left(-\mathbf{k}'\right) &= \frac{\sqrt{2\omega'}}{\left(2\pi\right)^{3/2}}\int\varphi\left(\mathbf{x},0\right)e^{i\mathbf{k}'\cdot\mathbf{x}}d^3x\\
a\left(\mathbf{k}'\right)-a^\dagger\left(-\mathbf{k}'\right) &= -\frac{i}{\omega'}\frac{\sqrt{2\omega'}}{\left(2\pi\right)^{3/2}}\int\pi\left(\mathbf{x},0\right)e^{i\mathbf{k}'\cdot\mathbf{x}}d^3x
\end{aligned}
$$

Adding gives $a(\mathbf{k}')$:

$$
a(\mathbf{k}')=\frac{\sqrt{2\omega'}}{2\left(2\pi\right)^{3/2}}\int\left(\varphi\left(\mathbf{x},0\right)-\frac{i}{\omega'}\pi\left(\mathbf{x},0\right)\right)e^{i\mathbf{k}'\cdot\mathbf{x}}d^3x \tag{10}
$$

while subtracting then changing the sign of $\mathbf{k}'$ gives the adjoint:

$$
a^\dagger(\mathbf{k}')=\frac{\sqrt{2\omega'}}{2\left(2\pi\right)^{3/2}}\int\left(\varphi(\mathbf{x},0)+\frac{i}{\omega'}\pi(\mathbf{x},0)\right)e^{-i\mathbf{k}'\cdot\mathbf{x}}d^3x \tag{11}
$$

This gives the amplitudes in terms of the field and its conjugate momentum. So far, this result is classical.

4.2 Quantization of the amplitudes

Next, we check the consequences of quantization for the amplitudes. Clearly, once φ and π become operators, the amplitudes do too. From the commutation relations for φ and π we can compute those for a and $a^\dagger$.

$$
\left[\hat{a}\left(\mathbf{k}\right),\hat{a}^\dagger\left(\mathbf{k}'\right)\right]=\frac{\omega'}{2\left(2\pi\right)^3}\int e^{i\mathbf{k}\cdot\mathbf{x}}d^3x\int e^{-i\mathbf{k}'\cdot\mathbf{x}'}d^3x'\left[\hat{\varphi}\left(\mathbf{x}\right)-\frac{i}{\omega}\hat{\pi}\left(\mathbf{x}\right),\hat{\varphi}\left(\mathbf{x}'\right)+\frac{i}{\omega}\hat{\pi}\left(\mathbf{x}'\right)\right]
$$

We need the commutator

$$
\begin{aligned}
\left[\hat{\varphi}\left(\mathbf{x}\right)-\frac{i}{\omega}\hat{\pi}\left(\mathbf{x}\right),\hat{\varphi}\left(\mathbf{x}'\right)+\frac{i}{\omega}\hat{\pi}\left(\mathbf{x}'\right)\right] &= -\frac{2i}{\omega}\left[\hat{\pi}\left(\mathbf{x}\right),\hat{\varphi}\left(\mathbf{x}'\right)\right]\\
&= \frac{2}{\omega}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left[\hat{a}(\mathbf{k}),\hat{a}^\dagger(\mathbf{k}')\right] &= \frac{\sqrt{\omega\omega'}}{2\left(2\pi\right)^3}\int\int e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{k}'\cdot\mathbf{x}'}d^3x\ d^3x'\frac{2}{\omega}\delta^3(\mathbf{x}-\mathbf{x}')\\
&= \frac{1}{\left(2\pi\right)^3}\sqrt{\frac{\omega'}{\omega}}\int e^{i\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{x}}d^3x\\
&= \delta^3\left(\mathbf{k}-\mathbf{k}'\right)
\end{aligned}
$$

Notice that the delta function makes the frequencies equal, $\omega=\omega'$. The commutator is reminiscent of the raising and lowering operators of the simple harmonic oscillator, and serve a similar function.

Exercise: Show that $\left[\hat{a}\left(\mathbf{k}\right),\hat{a}\left(\mathbf{k}'\right)\right]=0$.

Exercise: Show that $\left[\hat{a}^\dagger\left(\mathbf{k}\right),\hat{a}^\dagger\left(\mathbf{k}'\right)\right]=0$.

Finally, we summarize by the field and momentum density operators in terms of the mode amplitude operators:

$$\hat{\varphi}(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} \left(\hat{a}(\mathbf{k})\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\mathbf{k})\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \right) \tag{12}$$

$$\hat{\pi}(\mathbf{x},t) = \frac{i}{(2\pi)^{3/2}} \int \sqrt{\frac{\omega}{2}} d^3k \left(\hat{a}(\mathbf{k})\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} - \hat{a}^\dagger(\mathbf{k})\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \right) \tag{13}$$

Next, we turn to a study of states. We will begin with the Hamiltonian operator, which requires a bit of calculation.