Spinors and the Dirac equation

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When we work with linear representations of Lie groups and Lie algebras, it is important to keep track of the objects on which the operators act. These objects are always the elements of a vector space. In the case of $O(3)$, the vector space is Euclidean 3-space, while for Lorentz transformations the vector space is spacetime. As we shall see in this section, the covering groups of these same symmetries act on other, more abstract, complex vector spaces. The elements of these complex vector spaces are called spinors.

1 Spinors for $O(3)$

Let's start with $O(3)$, the group which preserves the lengths, $\mathbf{x}^2 = x^2 + y^2 + z^2 = g_{ij}x^i x^j$ of Euclidean 3-vectors. We can encode this length as the determinant of a matrix,

$$\begin{aligned} X &= \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \\ \det X &= -\left(x^2 + y^2 + z^2\right) \end{aligned}$$

This fact is useful because matrices of this type are easy to characterize. Let

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be any matrix with complex entries and demand hermiticity, $M = M^\dagger$:

$$\begin{aligned} M &= M^\dagger \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \end{aligned}$$

Then $\alpha \to a$ is real, $\delta \to d$ is real, and $\beta = \gamma^*$. Only $\gamma = b + ic$ remains arbitrary. If we also require M to be traceless, then M reduces to

$$M = \begin{pmatrix} a & b - ic \\ b + ic & -a \end{pmatrix}$$

just the same as X. Therefore, rotations may be characterized as the set of transformations of $X \to X'$ preserving the following properties of X :

1. Determinant: $\det X' = \det X$
2. Hermiticity: $X^\dagger = X$
3. Tracelessness: $tr\,(X) = 0$

To find the set of such transformations, recall that matrices transform by a similarity transformation

$$X \to X' = AXA^\dagger$$

Here we use the adjoint instead of the inverse because we imagine X to be doubly covariant, X_{ij}. For the mixed form, $X^{i}{}_{j}$ we would write $X \to AXA^{-1}$). From this form, we have:

$$\begin{aligned} \det X' &= \det \left(AXA^{\dagger}\right) \\ &= (\det A)(\det X)\left(\det A^{\dagger}\right) \end{aligned}$$

so we demand

$$\begin{aligned} |\det A|^2 &= 1 \\ \det A &= e^{i\varphi} \end{aligned}$$

We can constrain this determinant further, because if we write $A = e^{i\varphi/2}U$ where $\det U = 1$, then

$$\begin{aligned} X' &= AXA^{\dagger} \\ &= e^{i\varphi/2}UXe^{-i\varphi/2}U^{\dagger} \\ &= UXU^{\dagger} \end{aligned}$$

This means that a pure phase has no effect on X, so without loss of generality (and as required by uniqueness) we take the determinant of A to be one.

Next, notice that hermiticity is automatic. Whenever X is hermitian we have

$$\begin{aligned} (X')^{\dagger} &= \left(AXA^{\dagger}\right)^{\dagger} \\ &= A^{\dagger\dagger}X^{\dagger}A^{\dagger} \\ &= AXA^{\dagger} \\ &= X' \end{aligned}$$

so X' is hermitian.

Finally, we impose the trace condition. Suppose $tr(X) = 0$. Then

$$\begin{aligned} tr(X') &= tr(AXA^{\dagger}) \\ &= tr(A^{\dagger}AX) \end{aligned}$$

For the final expression to reduce to $tr(X)$ for all X, we must have $A^{\dagger}A = 1$. Therefore, $A^{\dagger} = A^{-1}$ and the transformations must be unitary. Using the unit determinant unitary matrices, U, we see that the group is $SU(2)$. This shows that $SU(2)$ can be used to write 3-dimensional rotations. In fact, we will see that $SU(2)$ includes two transformations corresponding to each element of $SO(3)$.

The exponential of any anti-hermitian matrix is unitary matrix because if $U = \exp(iH)$ with $H^{\dagger} = H$, then

$$U^{\dagger} = \exp\left(-iH^{\dagger}\right) = \exp(-iH) = U^{-1}$$

Conversely, any unitary matrix may be written this way. Moreover, since $\det A = e^{tr(\ln A)}$ the transformation $U = \exp(iH)$ has unit determinant whenever H is traceless. Since every traceless, hermitian matrix is a linear combination of the Pauli matrices,

$$\sigma_m = \left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right) \tag{1}$$

we may write every element of $SU(2)$ as the exponential

$$U(w^m) = e^{iw^m \sigma_m}$$

where the three parameters w^m are real and the Pauli matrices are mixed type tensors, $\sigma_m = [\sigma_m]^{a}{}_{b}$, because U is a transformation matrix.

There is a more convenient way to collect the real parameters w^m. Define a unit vector $\hat{\mathbf{n}}$ so that

$$\mathbf{w} = \frac{\varphi}{2}\hat{\mathbf{n}}$$

Then

$$U(\varphi, \hat{\mathbf{n}}) = \exp\left(\frac{i\varphi}{2}\hat{\mathbf{n}} \cdot \sigma\right) \tag{2}$$

is a rotation through an angle φ about the $\hat{\mathbf{n}}$ direction.

Exercise: Let $\hat{\mathbf{n}} = (0, 0, 1)$ and show that the relation between (x, y, z) and (x', y', z') given by

$$\begin{aligned} X' &= \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix} = UXU^\dagger \\ &= \exp\left(\frac{i\varphi}{2}\hat{\mathbf{n}} \cdot \sigma\right) \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \exp\left(-\frac{i\varphi}{2}\hat{\mathbf{n}} \cdot \sigma\right) \end{aligned}$$

is a rotation by φ about the z axis.

Exercise: By expanding the exponential in a power series and working out the powers of $\hat{\mathbf{n}} \cdot \sigma$ for a general unit vector $\hat{\mathbf{n}}$, prove the identity

$$\exp\left(\frac{i\varphi}{2}\hat{\mathbf{n}} \cdot \sigma\right) = \mathbf{1}\cos\frac{\varphi}{2} + i\hat{\mathbf{n}} \cdot \sigma \ \sin\frac{\varphi}{2} \tag{3}$$

Also, show that $U(2\pi, \hat{\mathbf{n}}) = -1$ and $U(4\pi, \hat{\mathbf{n}}) = 1$ for any unit vector, $\hat{\mathbf{n}}$. From this, show that $U(2\pi, \hat{\mathbf{n}})$ gives $X' = X$.

Now let's consider what vector space $SU(2)$ acts on. We have used a similarity transformation on matrices to show how it acts on a 3-dimensional subspace of the 8-dimensional space of 2×2 complex matrices. But more basically, $SU(2)$ acts the vector space of complex, two component *spinors*:

$$\begin{aligned} \chi &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \chi' &= U\chi \end{aligned}$$

Exercise: Using the result of the previous exercise, eq.(3) find the most general action of $SU(2)$ on χ. Show that the periodicity of the mapping is 4π, that is, that

$$U(4\pi m, \hat{\mathbf{n}})\chi = \chi$$

for all integers m, while $U(2\pi m, \hat{\mathbf{n}})\chi = -\chi \neq \chi$ for odd m.

The vector space of spinors χ is the simplest set of objects that Euclidean rotations act nontrivially on. These objects are familiar from quantum mechanics as the spin-up and spin-down states of spin-1/2 fermions. It is interesting to observe that spin is a perfectly classical property arising from symmetry. It was not necessary to discover quantum mechanics in order to discover spin. Apparently, the reason that "classical spin" was not discovered first is that its magnitude is microscopic. Indeed, with the advent of supersymmetry, there has been some interest in classical supersymmetry – supersymmetric classical theories whose quantization leads to now-familiar quantum field theories.

2 Spinors for the Lorentz group

Next, we extend this new insight to the Lorentz group. Recall that we defined Lorentz transformations as those preserving the Minkowski line element,

$$s^2 = t^2 - (x^2 + y^2 + z^2)$$

or equivalently, those transformations leaving the Minkowski metric invariant. Once again, we write a matrix that contains the invariant information in its determinant. Let

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$$

noting that X is now the most general hermitian 2×2 matrix, $X^\dagger = X$, without any constraint on the trace. The determinant is now

$$\det X = t^2 - x^2 - y^2 - z^2 = s^2$$

and we only need to preserve two properties: Hermiticity and unit determinant.

Let $X' = AXA^\dagger$. Then Hermiticity is again automatic and all we need is $|\det A|^2 = 1$. As before, an overall phase does not affect X, so we can fix $\det A = 1$. There is no further constraint needed, so Lorentz transformations is given by the special linear group in two complex dimensions, $SL(2, \mathbb{C})$. Let's find the generators. First, it is easy to find a set of generators for the general linear group, because every non-degenerate matrix is allowed. Expanding a general matrix infinitesimally about the identity gives

$$G = 1 + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for small-normed complex numbers $\alpha, \beta, \gamma, \delta$. Since the deviation from the identity is small, the determinant will be close to one. Dropping quadratic terms, we explicitly compute,

$$\begin{aligned} 1 &= \det G \\ &= \det \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix} \\ &= (1+\alpha)(1+\delta) - \beta\gamma \\ &\approx 1 + \alpha + \delta \end{aligned}$$

so the unit determinant is achieved by making the generators traceless, setting $\delta = -\alpha$. A complete set of generators for $SL(2, \mathbb{C})$ is therefore

$$\begin{aligned} K^i \equiv \sigma^i &= \left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix} \right] \\ J^i \equiv i\sigma^i &= \left[\begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} & i \\ i & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right] \end{aligned}$$

Because any six independent linear combinations of these are an equivalently good basis, let's choose the K^i, J^i, which have the advantage of being hermitian and anti-hermitian, respectively.

When we exponentiate J_m and K_m (with real parameters) to recover the various types of Lorentz transformation, the anti-hermitian generators J_m give $SU(2)$ as before. We already know that these preserve lengths of spatial 3-vectors, so we see again that the 3-dimensional rotations are part of the Lorentz group. Since the generators K_m are hermitian, the corresponding group elements are not unitary. The corresponding transformations are hyperbolic rather than circular, corresponding to boosts.

Exercise: Recalling the Taylor series

$$\begin{aligned}\sinh\lambda &= \sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{(2k+1)!}\\ \cosh\lambda &= \sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k)!}\end{aligned}$$

show that $K_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ generates a boost in spacetime.

The Lie algebra of $SL(2,\mathbb{C})$ is now easy to calculate. Since the Pauli matrices multiply as (exercise!

$$\sigma_m\sigma_n = \delta_{mn}\mathbf{1} + i\varepsilon_{mnk}\sigma_k \tag{4}$$

their commutators are $[\sigma_m,\sigma_n] = 2i\varepsilon_{mnk}\sigma_k$, we compute the the Lie algebra, identifying the resultant generator as J_i or K_i so as to keep the structure constants real. Thus,

$$\begin{aligned}[J_m,J_n] &= [i\sigma_m,i\sigma_n]\\ &= -2i\varepsilon_{mnk}\sigma_k\\ &= -2\varepsilon_{mnk}(i\sigma_k)\\ &= -2\varepsilon_{mnk}J_k\\ [J_m,K_n] &= [i\sigma_m,\sigma_n]\\ &= i(2i\varepsilon_{mnk}\sigma_k)\\ &= -2\varepsilon_{mnk}K_k\\ [K_m,J_n] &= [\sigma_m,i\sigma_n]\\ &= i(2i\varepsilon_{mnk}\sigma_k)\\ &= -2\varepsilon_{mnk}K_k\\ [K_m,K_n] &= [\sigma_m,\sigma_n]\\ &= 2i\varepsilon_{mnk}\sigma_k\\ &= 2\varepsilon_{mnk}J_k\end{aligned}$$

gives the complete Lie algebra of the Lorentz group,

$$\begin{aligned}[J_m,J_n] &= -2\varepsilon_{mnk}J_k\\ [J_m,K_n] &= -2\varepsilon_{mnk}K_k\\ [K_m,J_n] &= -2\varepsilon_{mnk}K_k\\ [K_m,K_n] &= 2\varepsilon_{mnk}J_k\end{aligned} \tag{5}$$

This is an important result. It shows that while the rotations form a subgroup of the Lorentz group (because the J_m commutators close into themselves), the boosts do not – two boosts applied in succession produce a rotation as well as a change of relative velocity. This is the source of a noted correction to angular momentum, the Thomas precession (see Jackson, pp. 556-560; indeed, see Jackson's chapters 11 and 12 for a good discussion of special relativity in a context with real examples).

There is another convenient basis for the Lorentz Lie algebra. Consider the six generators

$$\begin{aligned}L_m &= \frac{1}{2}(J_m + iK_m)\\ M_m &= \frac{1}{2}(J_m - iK_m)\end{aligned}$$

These satisfy

$$
\begin{aligned}
[L_m, L_n] &= \left[\frac{1}{2}(J_m + iK_m), \frac{1}{2}(J_n + iK_n)\right] \\
&= \frac{1}{4}([J_m, J_n] + i[J_m, K_n] + i[K_m, J_n] - [K_m, K_n]) \\
&= \frac{1}{4}(-2\varepsilon_{mnk}J_k - 2i\varepsilon_{mnk}K_k - 2\varepsilon_{nmk}iK_k - 2\varepsilon_{mnk}J_k) \\
&= -\varepsilon_{mnk}(J_k + iK_k) \\
&= -\varepsilon_{mnk}L_k \\
[L_m, M_n] &= \left[\frac{1}{2}(J_m + iK_m), \frac{1}{2}(J_n - iK_n)\right] \\
&= \frac{1}{4}([J_m, J_n] - i[J_m, K_n] + i[K_m, J_n] + [K_m, K_n]) \\
&= \frac{1}{4}(-2\varepsilon_{mnk}J_k + 2i\varepsilon_{mnk}K_k - 2i\varepsilon_{nmk}K_k + 2\varepsilon_{mnk}J_k) \\
&= 0 \\
[M_m, M_n] &= \left[\frac{1}{2}(J_m - iK_m), \frac{1}{2}(J_n - iK_n)\right] \\
&= \frac{1}{4}([J_m, J_n] - i[J_m, K_n] - i[K_m, J_n] - [K_m, K_n]) \\
&= \frac{1}{4}(-2\varepsilon_{mnk}J_k + 2i\varepsilon_{mnk}K_k + 2i\varepsilon_{nmk}K_k - 2\varepsilon_{mnk}J_k) \\
&= -\varepsilon_{mnk}(J_k - iK_k) \\
&= -\varepsilon_{mnk}M_k
\end{aligned}
$$

showing that the Lorentz group actually decouples into two commuting copies of $SU(2)$. Extensive use of this fact is made in general relativity (see, eg., Penrose and Rindler, Wald). In particular, we can use this decomposition of the Lie algebra $sl(2, C)$ to introduce *two* sets of 2-component spinors, called *Weyl spinors*,

$$\chi^A, \bar{\chi}^{\dot{A}}$$

with the first set transforming under the action of $\exp(u^m L_m)$ and the second set under $\exp(v^m M_m)$. For our study of field theory, however, we will be more interested in a different set of spinors – the 4-component *Dirac spinors.*

3 Dirac spinors and the Dirac equation

There is a systematic way to develop spinor representations of any pseudo-orthogonal group, $O(p, q)$. However, Dirac arrived at this representation when he sought a relativistic form for quantum theory. We won't look at the full historical rationale for Dirac's approach, but will use a similar construction. Dirac wanted to build a relativistic quantum theory, and recognizing that relativity requires space and time variables to enter on the same footing, sought an equation *linear* in both space and time derivatives:

$$i\frac{\partial\psi}{\partial t} = \left(-i\alpha^i \partial_i + m\beta\right)\psi \tag{6}$$

where the γ^μ and β are constant. A quadratic equation, the Klein-Gordon equation,

$$\Box\phi = -m^2\phi \tag{7}$$

had already been tried and discarded by Schrödinger because the second order equation requires two initial conditions and the uncertainty principle allows us only one. To determine the coefficients, Dirac demanded that the linear equation should imply the Klein-Gordon equation. Acting on our version of Dirac's equation with the same operator again,

$$\begin{aligned}-\frac{\partial^2\psi}{\partial t^2} &= \left(-i\alpha^i\partial_i + m\beta\right)\left(-i\alpha^i\partial_i + m\beta\right)\psi \\ &= \left(-\alpha^i\alpha^j\partial_i\partial_j - im\alpha^i\beta\partial_i - im\beta\alpha^i\partial_i + m^2\beta^2\right)\psi\end{aligned}$$

we reproduce the Klein-Gordon equation provided

$$\begin{aligned}-\alpha^i\alpha^j\partial_i\partial_j &= -\nabla^2 \\ m\left(\alpha^i\beta + \beta\alpha^i\right)\partial_i &= 0 \\ m^2\beta^2 &= m^2\end{aligned}$$

or equivalently,

$$\begin{aligned}\alpha^i\alpha^j + \alpha^i\alpha^i &= 2\delta^{ij} \\ \alpha^i\beta + \beta\alpha^i &= 0 \\ \beta^2 &= 1\end{aligned}$$

We can put these conditions into a more systematic and relativistic form by defining

$$\gamma^\mu = \left(\beta, \beta\alpha^i\right)$$

Then the constraints on γ^μ become

$$\begin{aligned}\gamma^i\gamma^j + \gamma^j\gamma^i &= -2\delta^{ij} \\ \gamma^i\gamma^0 + \gamma^0\gamma^i &= 0 \\ \left(\gamma^0\right)^2 &= 1\end{aligned}$$

which may be neatly expressed as

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{8}$$

where the curly brackets denote the anti-commutator. This relationship is impossible to achieve if γ^μ is a vector. To see this, note that we can always perform a Lorentz transformation that brings γ^μ to one of the forms

$$\begin{aligned}\gamma^\mu &= (\alpha, 0, 0, 0) \\ \gamma^\mu &= (\alpha, \alpha, 0, 0) \\ \gamma^\mu &= (0, \alpha, 0, 0)\end{aligned}$$

depending on whether γ^μ is timelike, null or spacelike. Then, since $\eta^{\mu\nu}$ is Lorentz invariant, we have the possibilities:

$$\begin{aligned}\{\gamma^\mu, \gamma^\nu\} &= \begin{pmatrix}\alpha^2 &&& \\ & 0 && \\ && 0 & \\ &&& 0\end{pmatrix} \\ \{\gamma^\mu, \gamma^\nu\} &= \begin{pmatrix}\alpha^2 & \alpha^2 && \\ \alpha^2 & \alpha^2 && \\ && 0 & \\ &&& 0\end{pmatrix} \\ \{\gamma^\mu, \gamma^\nu\} &= \begin{pmatrix}0 &&& \\ & \alpha^2 && \\ && 0 & \\ &&& 0\end{pmatrix}\end{aligned}$$

none of which equals $\eta^{\mu\nu}$. Therefore, γ^μ must be a more general kind of object. It *is* sufficient to let γ^μ be a set of four, 4×4 matrices, and it is not hard to show that this is the smallest size matrix that works.

Exercise: Show that there do not exist four, 2×2 matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Here is a convenient choice for the *Dirac matrices*, or *gamma matrices*:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \end{aligned} \tag{9}$$

where the σ^i are the usual 2×2 Pauli matrices.

Exercise: Show that these matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Substituting γ^μ into eq.(6), we have the *Dirac equation*,

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{10}$$

This equation gives us more than we bargained for. Since the γ^μ are 4×4 Dirac matrices, the object ψ that they act on must also be a 4-component vector. The complex vector ψ transforms as a spinor representation of the Lorentz group.

4 The Dirac action

The Dirac equation is the field equation for a spin-$\frac{1}{2}$ field. Since we have an invariant inner product, we can write an invariant action as

$$S = \int d^4x\ \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi \tag{11}$$

The action is to be varied with respect to ψ and $\bar{\psi}$ independently

$$0 = \delta S = \int d^4x\ \left(\delta\bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi + \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\delta\psi\right)$$

The $\bar{\psi}$ variation immediately yields the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

while the $\delta\psi$ requires integration by parts,

$$\begin{aligned} 0 &= \int d^4x\ \bar{\psi}\,(i\gamma^\mu \partial_\mu \delta\psi - m\delta\psi) \\ &= \int d^4x\ \left(-i\partial_\mu \bar{\psi}\gamma^\mu \delta\psi - \bar{\psi}m\right)\delta\psi \end{aligned}$$

Thus

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

which is sometimes written as

$$\bar{\psi}\left(i\gamma^\mu \overleftarrow{\partial}_\mu + m\right) = 0$$

This is the conjugate Dirac equation.

5 Casimir Operators

Casimir operators give us a way to classify representations of groups.

For any Lie algebra, $\mathcal{G}$, with generators G_a and commutators

$$[G_a, G_b] = c_{ab}{}^{c} G_c$$

we can consider composite operators found by multiplying together two or more generators, $G_1G_2, G_3G_9G_{17}, \ldots$ and taking linear combinations,

$$A = \alpha G_1 G_2 + \beta G_3 G_9 G_{17} + \ldots$$

The set of all such linear combinations of products is called the *free algebra* of $\mathcal{G}$. Among the elements of the free algebra are a very few special cases called *Casmir operators*, which have the special property of commuting with all of the generators. For example; the generators J_i of $O(3)$ may be combined into the combination

$$\vec{J}^2 = \delta^{ij} J_i J_j = \sum (J_i)^2 \tag{12}$$

We can compute

$$\begin{aligned}
\left[J_i, \vec{J}^2\right] &= \left[J_i, \sum (J_j)^2\right] \\
&= \sum \left(J_j \left[J_i, J_j\right] + \left[J_i, J_j\right] J_j\right) \\
&= \sum \left(J_j \varepsilon_{ijk} J_k + \varepsilon_{ijk} J_k J_j\right) \\
&= \varepsilon_{ijk} \left(J_j J_k + J_k J_j\right) \\
&= 0
\end{aligned}$$

where, in the last step, we used the fact that ε_{ijk} is antisymmetric on jk, while the expression $J_jJ_k + J_kJ_k$ is explicitly symmetric. R is therefore a Casimir operator for $O(3)$. Notice that since R commutes with all of the generators, it must also commute with all elements of $O(3)$ (*Exercise!!*). For this reason, Casimir operators become extremely important in quantum physics. Because the symmetries of our system are group symmetries, the set of all Casimir operators gives us a list of the conserved quantities. Often, elements of a Lie group take us from one set of fields to a physically equivalent set. Since the Casimir operators are left invariant, we can use eigenvalues of the Casimir operators to classify the possible distinct physical states of the system.

Let's look at the Casimir operators that are most important for particle physics – those of the Poincaré group. The Poincaré group is the set of transformations leaving the *infinitesimal* line element

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{13}$$

invariant. It clearly includes Lorentz transformations,

$$\left[dx'\right]^\alpha = \Lambda^\alpha{}_\beta dx^\beta$$

but now also includes translations:

$$\begin{aligned}
\left[x'\right]^\alpha &= x^\alpha + a^\alpha \\
&\Rightarrow \left[dx'\right]^\alpha = dx^\beta
\end{aligned}$$

Since there are 4 translations and 6 Lorentz transformations, there are a total of 10 Poincaré symmetries. There are several ways to write a set of generators for these transformations. One common one is to let

$$\begin{aligned}
M^\alpha{}_\beta &= x^\alpha \partial_\beta - x_\beta \partial^\alpha \\
P_\alpha &= \partial_\alpha
\end{aligned} \tag{14}$$

Then it is easy to show that

$$\begin{aligned}
\left[M^{\alpha}{}_{\beta}, M^{\mu}{}_{\nu}\right] &= \left[x^{\alpha}\partial_{\beta} - x_{\beta}\partial^{\alpha}, x^{\mu}\partial_{\nu} - x_{\nu}\partial^{\mu}\right] \\
&= x^{\alpha}\partial_{\beta}\left(x^{\mu}\partial_{\nu} - x_{\nu}\partial^{\mu}\right) - x_{\beta}\partial^{\alpha}\left(x^{\mu}\partial_{\nu} - x_{\nu}\partial^{\mu}\right) \\
&\quad - x^{\mu}\partial_{\nu}\left(x^{\alpha}\partial_{\beta} - x_{\beta}\partial^{\alpha}\right) + x_{\nu}\partial^{\mu}\left(x^{\alpha}\partial_{\beta} - x_{\beta}\partial^{\alpha}\right) \\
&= x^{\alpha}\delta^{\mu}_{\beta}\partial_{\nu} - x^{\alpha}\eta_{\beta\nu}\partial^{\mu} - x_{\beta}\eta^{\alpha\mu}\partial_{\nu} + x_{\beta}\delta^{\alpha}_{\nu}\partial^{\mu} \\
&\quad - x^{\mu}\delta^{\alpha}_{\nu}\partial_{\beta} + x^{\mu}\eta_{\nu\beta}\partial^{\alpha} + x_{\nu}\eta^{\mu\alpha}\partial_{\beta} - x_{\nu}\delta^{\mu}_{\beta}\partial^{\alpha} \\
&= \delta^{\mu}_{\beta}M^{\alpha}{}_{\nu} - \eta_{\beta\nu}M^{\alpha\mu} - \eta^{\alpha\mu}M_{\beta\nu} + \delta^{\alpha}_{\nu}M_{\beta}{}^{\mu} \qquad (15)
\end{aligned}$$

To compute these, we imagine the derivatives acting on a function to the right of the commutator, $\left[M^{\alpha}{}_{\beta}, M^{\mu}{}_{\nu}\right] f(x)$. Then all of the derivatives of f cancel when we antisymmetrize. Two similar but shorter calculations show that

$$\begin{aligned}
\left[M^{\alpha}{}_{\beta}, P_{\nu}\right] &= \eta_{\nu\beta}P^{\alpha} - \delta^{\alpha}_{\nu}P_{\beta} \qquad (16) \\
\left[P_{\alpha}, P_{\beta}\right] &= 0 \qquad (17)
\end{aligned}$$

Eqs.(15-17) comprise the Lie algebra of the Poincaré group.

Exercise: Prove eq.(16) and eq.(17) using eqs.(14).

Exercise: Show, by a suitable identifications of the generators, that the general form of the Lie algebra case of $SO(p,q)$ eq.(15) agrees with the form found for the Lorentz group $SO(1,3)$, eq.(5).

Now we can write the Casimir operators of the Poincaré group. There are two,

$$\begin{aligned}
P^2 &= \eta^{\alpha\beta}P_{\alpha}P_{\beta} \\
W^2 &= \eta_{\alpha\beta}W^{\alpha}W^{\beta}
\end{aligned}$$

where

$$W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_{\nu}M_{\alpha\beta}$$

and $\varepsilon^{\mu\nu\alpha\beta}$ is the spacetime Levi-Civita tensor.

To see what these correspond to, recall from our discussion of Noether currents that the conservation of 4-momentum is associated with translation invariance, and P_{α} is the generator of translations. In fact, $P_{\alpha} = i\partial_{\alpha}$, the Hermitian form of the translation generator, is the usual energy-momentum operator of quantum mechanics. We directly interpret eigenvectors of P_{α} as energy and momentum. Thus, we expect that eigenvalues of P^2 will be the mass, $P_{\alpha}P^{\alpha} = m^2$.

If we boost P_{α} to it's rest frame, $P_{\alpha} = m(c, \mathbf{0})$, then

$$\begin{aligned}
W^{\mu} &= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_{\nu}M_{\alpha\beta} \\
&= \frac{1}{2}mc\varepsilon^{\mu0\alpha\beta}M_{\alpha\beta} \\
&= -\frac{1}{2}mc\varepsilon^{0\mu\alpha\beta}M_{\alpha\beta}
\end{aligned}$$

In this frame, $W^{\mu} = \left(0, W^{i}\right)$ where

$$\begin{aligned}
W^{i} &= \frac{1}{2}mc\varepsilon^{ijk}M_{jk} \\
&= mcJ^{i}
\end{aligned}$$

where the J^i are the generators of $SO(3)$. Then

$$W^\mu W_\mu = m^2 c^2 J^2$$

and since m^2 is separately conservec, we see that W^2 is the Casimir of $SO(3)$, the conserved total angular momentum.

Exercise: Using the Lie algebra of the Poincaré group, eqs.(15-17), prove that P^2 and W^2 commute with $M_{\alpha\beta}$ and P_α. (Warning! The proof for W^2 is a bit tricky!) Notice that the proof requires only the Lie algebra relations for the Poincaré group, and *not* the specific representation of the operators given in eqs.(14).

Since the Casimir operators of the Poincaré group correspond to mass and spin, we will be able to classify states of quantum fields by mass and spin. We will extend this list when we introduce additional symmetry groups.

Consider the action of W^2 on a Dirac spinor. Acting with $W^\mu W_\mu$,

$$\frac{1}{m^2 c^2} W^\mu W_\mu \psi = J^2 \psi$$

6 Further properties of the Dirac matrices

In four dimensions, there are 16 independent matrices that we can construct from the Dirac matrices. We have already encountered eleven of them:

$$\mathbf{1}, \gamma^\mu, \sigma^{\mu\nu}$$

The remaining five are most readily expressed in terms of

$$\gamma_5 \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3$$

Exercise: Prove that γ_5 is hermitian.

Exercise: Prove that $\{\gamma_5, \gamma^\mu\} = 0$.

Exercise: Prove that $\gamma_5 \gamma_5 = 1$.

Then the remaining five matrices may be taken as

$$\gamma_5, \gamma_5 \gamma^\mu$$

Any 4×4 matrix can be expressed as linear combination of these 16 matrices. We will need several other properties of these matrices. First, if we contract the product of pair of gammas, we get 4,

$$\gamma^\mu \gamma_\mu = \eta_{\mu\nu} \gamma^\mu \gamma^\nu = \frac{1}{2} \eta_{\mu\nu} \{\gamma^\mu, \gamma^\nu\} = \eta_{\mu\nu} \eta^{\mu\nu} = 4$$

We also need various traces. For any product of an odd number of gamma matrices we have

$$\begin{aligned} tr\left(\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_{2k+1}}\right) &= tr\left(\left(\gamma_5\right)^2 \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_{2k+1}}\right) \\ &= (-1)^{2k+1}\, tr\left(\gamma_5 \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_{2k+1}} \gamma_5\right) \end{aligned}$$

using the fact that γ_5 commutes with any of the γ^μ. Now, using the cyclic property of the trace

$$tr\,(A \ldots BC) = tr\,(CA \ldots B)$$

we cycle γ_5 back to the front:

$$\begin{aligned} tr\left(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_{2k+1}}\right) &= (-1)^{2k+1}\ tr\left(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_{2k+1}}\gamma_5\right) \\ &= (-1)^{2k+1}\ tr\left(\gamma_5\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_{2k+1}}\right) \\ &= -\ tr\left(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_{2k+1}}\right) \\ &= 0 \end{aligned}$$

Thus, the trace of the product of any *odd* number of gamma matrices vanishes.

Traces of even numbers are trickier. For two:

$$\begin{aligned} tr\left(\gamma^{\mu}\gamma^{\nu}\right) &= tr\left(-\gamma^{\nu}\gamma^{\mu} + 2\eta^{\mu\nu}\mathbf{1}\right) \\ &= -tr\left(\gamma^{\nu}\gamma^{\mu}\right) + 2\eta^{\mu\nu} tr\ \mathbf{1} \end{aligned}$$

or, since $tr\mathbf{1} = 4$,

$$tr\left(\gamma^{\mu}\gamma^{\nu}\right) = 4\eta^{\mu\nu}$$

Exercise: Prove that

$$tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\right) = 4\left(\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}\right)$$

Exercise: Prove that

$$\gamma^{\mu}\gamma^{\alpha}\gamma_{\mu} = -2\gamma^{\alpha}$$

and

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\gamma_{\mu} = 4\eta^{\alpha\beta}$$