Special Relativity

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Since we have just introduced some relativitistic notation, this seems like a good place to review special relativity, and especially the reason that the notation is meaningful.

1 The invariant interval

The first thing to understand clearly is the difference between physical quantities such as the length of a ruler or the elapsed time on a clock, and the coordinates we use to label locations in the world. In 3-dim Euclidean geometry, for example, the length of a ruler is given in terms of coordinate intervals using the Pythagorean theorem. Thus, if the positions of the two ends of the ruler are (x_1, y_1, z_1) and (x_2, y_2, z_2), the length is

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Observe that the actual values of (x_1, y_1, z_1) are irrelevant. Sometimes we choose our coordinates cleverly, say, by aligning the x-axis with the ruler and placing one end at the origin so that the endpoints are at $(0,0,0)$ and $(x_2, 0, 0)$. Then the *calculation* of L is trivial

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = x_2$$

but it is still important to recognize the difference between the coordinates and the length.

With this concept clear, we next need a set of labels for spacetime. Starting with a blank page to represent spacetime, we start to construct a set of labels. First, since all observers agree on the motion of light, let's agree that (with time flowing roughly upward in the diagram and space extending left and right) light beams always move at 45 degrees in a straight line. An inertial observer (whose constant rate of motion has no absolute reality; we only consider the relative motions of two observers) will move in a straight line at a steeper angle than 45 degrees – a lesser angle would correspond to motion faster than the speed of light. For any such inertial observer, we let the time coordinate be the time as measured by a clock they carry. The ticks of this clock provide a time scale along the straight, angled world line of the observer. To set spatial coordinates, we use the constancy of the speed of light. Suppose our inertial observer send out a pulse of light at 3 minutes before noon, and suppose the nearby spacetime is dusty enough that bits of that pulse are reflected back continuously. Then some reflected light will arrive back at the observer at 3 minutes after noon. Since the trip out and the trip back must have taken the same length of time and occurred with the light moving at constant velocity, the reflection of the light by the dust particle must have occurred at noon in our observer's frame of reference. It must have occurred at a distance of 3 light minutes away. If we take the x direction to be the direction the light was initially sent, the location of the dust particle has coordinates $(noon, 3\, light\, minutes, 0, 0)$. In a similar way, we find the locus of all points with time coordinate $t = noon$ and both $y = 0$ and $z = 0$. These points form our x axis. We find the y and z axes in the same way. It is somewhat startling to realize when we draw a careful diagram of this construction, that the x axis seems to make an acute angle with the time axis, as if the time axis has been reflected about the 45 degree path of a light beam. We quickly notice that this must always be the case if all observers are to measure the same speed ($c = 1$ in our construction) for light.

This gives us our labels for spacetime *events*. Any other set of labels would work just as well. In particular, we are interested in those other sets of coordinates we get by choosing a different initial world line of an different inertial observer. Suppose we consider two inertial observers moving with *relative* velocity v. Using such devices as mirror clocks and other thought experiments, most elementary treatments of special relativity quickly arrive at the relationship between such a set of coordinates. If the relative motion is in the x direction, the transformation between the two frames of reference is the familiar Lorentz transformation

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The next step is the most important: we must find a way to write physically meaningful quantities. These quantities, like length in Euclidean geometry, must be independent of the labels, the coordinates, that we put on different points. If we get on the right track by forming a quadratic expression similar to the Pythagorean theorem, then it doesn't take long to arrive at the correct answer. In spacetime, we have a *pseudo*-Euclidean length interval, given by

$$c^2\tau^2 = c^2t^2 - x^2 - y^2 - z^2 \tag{1}$$

Computing the same quantity in the primed frame, we find

$$\begin{aligned} c^2\tau'^2 &= c^2t'^2 - x'^2 - y'^2 - z'^2 \\ &= c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 - \gamma^2\left(x - vt\right)^2 - y^2 - z^2 \\ &= c^2\gamma^2\left(t^2 - \frac{2vxt}{c^2} + \frac{v^2x^2}{c^4}\right) \\ &\quad -\gamma^2\left(x^2 - 2xvt + v^2t^2\right) - y^2 - z^2 \\ &= \gamma^2\left(c^2t^2 - v^2t^2 - x^2 + \frac{v^2x^2}{c^2}\right) - y^2 - z^2 \\ &= c^2t^2 - x^2 - y^2 - z^2 \\ &= c^2\tau^2 \end{aligned}$$

so that $\tau = \tau'$. Tau is called the proper time, and is invariant under Lorentz transformations. It plays the role of L in spacetime geometry, and becomes the defining property of spacetime symmetry: *we define Lorentz transformations to be those transformations that leave τ invariant.*

2 Lorentz transformations

Notice that with this definition, 3-dim rotations are included as Lorentz transformations because τ only depends on the Euclidean length $x^2 + y^2 + z^2$; any transformation that leaves this length invariant also leaves τ invariant. Lorentz transformations that map the three spatial directions into one another are called rotations, while Lorentz transformations that involve time and velocity are called *boosts*. As we shall see, there are 6 independent Lorentz transformations: three planes $((xy), (yz), (zx))$ of rotation and three planes $((tx), (ty), (tz))$ of boosts.

Notice that Lorentz transformations are *linear*. If we define the 4×4 matrix

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & -\frac{v}{c^2} & & \\ -v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and the four coordinates by $x^{\alpha} = (ct, x, y, z)$, then a "boost in the x direction" is given by

$$(x')^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta}$$

where we assume a sum on β. Any object that transforms in this linear, homogeneous way, where $\Lambda^{\alpha}{}_{\beta}$ is any boost or rotation matrix, is called a *Lorentz vector* or a 4-*vector*. The *proper time*, or more generally, the *proper interval*, defines the allowed forms of $\Lambda^{\alpha}{}_{\beta}$; we say that $\Lambda^{\alpha}{}_{\beta}$ is the matrix of a Lorentz transformation if and only if it leaves all intervals invariant.

We can write the interval in terms of a *metric*. Let

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

as given in the previous section. Then the interval spanned by a 4-vector x^{α} is

$$c^2\tau^2 = \eta_{\alpha\beta} x^{\alpha} x^{\beta} = \begin{pmatrix} ct & x & y & z \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

It is convenient to define two different forms of a vector, called covariant (x_{α}) and contravariant (x^{α}). These two forms exist any time we have a metric. If we let

$$x_{\alpha} \equiv \eta_{\alpha\beta} x^{\beta}$$

then we can write invariant intervals as

$$c^2\tau^2 = x_{\beta} x^{\beta} = x^{\beta} x_{\beta}$$

where the second expression uses the symmetry of the metric, $\eta_{\alpha\beta} = \eta_{\beta\alpha}$.

The defining property of a Lorentz transformation can now be written in a way that doesn't depend on the coordinates. Invariance of the interval requires

$$c^2\tau^2 = \eta_{\alpha\beta} x^{\alpha} x^{\beta} = \eta_{\alpha\beta} (x')^{\alpha} (x')^{\beta}$$

so that for any Lorentz vector x^{β},

$$\begin{aligned} \eta_{\mu\nu} x^{\mu} x^{\nu} &= \eta_{\alpha\beta} (x')^{\alpha} (x')^{\beta} \\ &= \eta_{\alpha\beta} \left(\Lambda^{\alpha}{}_{\mu} x^{\mu}\right) \left(\Lambda^{\beta}{}_{\nu} x^{\nu}\right) \\ &= \left(\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}\right) x^{\mu} x^{\nu} \end{aligned}$$

Since x^{μ} is arbitrary, and $\eta_{\alpha\beta}$ is symmetric, this implies

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} \tag{2}$$

From now on, we will take this as the defining property of a Lorentz transformation.

Exercise: Prove that the Lorentz transformations form a group

Suppose w^α is any set of four quantities that transform just like the coordinates, so that if we boost or rotate to a new inertial frame, the new components of w^α are given by

$$(w')^\alpha = \Lambda^\alpha{}_\beta w^\beta$$

where $\Lambda^\alpha{}_\beta$ is the matrix describing the boost or rotation. It follows immediately that $w_\alpha w^\alpha$ is invariant under Lorentz transformations. As long as we are careful to use only quantities that have such simple transformations (i.e., linear and homogeneous) it is easy to construct Lorentz invariant quantities by "contracting" indices. Anytime we sum one contravariant vector index with one covariant vector index, we produce an invariant.

It is not hard to derive dynamical variables which are Lorentz vectors. Suppose we have a path in spacetime (perhaps the path of a particle), specified parametrically $x^\beta(\lambda)$ so as λ increases, $x^\beta(\lambda)$ gives the coordinates of the particle. We can even let λ be the proper time along the world line of the particle, since this increases monotonically as the particle moves along. In fact, this is an excellent choice. To compute the parameter, consider an infinitesimal displacement, dx^β, along the path. Then the change in the proper time for that displacement is

$$\begin{aligned} d\tau &= \left(\eta_{\alpha\beta} dx^\alpha dx^\beta\right)^{1/2} \\ &= \left(dt^2 - \frac{1}{c^2}\left(dx^i\right)^2\right)^{1/2} \end{aligned}$$

where the Latin index runs over the spatial coordinates so that $dx^i dx^i$ is the usual Euclidean interval. Now we can integrate the infinitesimal proper time along the path to a general point at proper time τ,

$$\begin{aligned} \tau &= \int d\tau \\ &= \int \sqrt{dt^2 - \frac{1}{c^2}\left(dx^i\right)^2} \\ &= \int dt \sqrt{1 - \frac{1}{c^2}\left(\frac{dx^i}{dt}\right)^2} \\ &= \int dt \sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}} \end{aligned}$$

As soon as we know the path $\mathbf{x}(t)$, we can differentiate to find $\mathbf{v}(t)$, integrate to find $\tau(t)$, and invert to find $t(\tau)$. This gives $x^\alpha(\tau) = (t(\tau), \mathbf{x}(\tau))$. Notice the useful relationship between infinitesimals, $d\tau = dt\sqrt{1 - \frac{\mathbf{v}^2(t)}{c^2}}$, or

$$\gamma d\tau = dt$$

Once we have the path parameterized in terms of proper time, we can find the tangent to the path simply by differentiating,

$$u^\beta = \frac{dx^\beta}{d\tau}$$

Since τ is Lorentz invariant and the Lorentz transformation matrix is constant (between two given inertial frames), we have

$$\begin{aligned} (u')^\beta &= \frac{d\,(x')^\beta}{d\tau'} \\ &= \frac{d\left(\Lambda^\beta{}_\alpha x^\alpha\right)}{d\tau} \\ &= \Lambda^\beta{}_\alpha u^\alpha \end{aligned}$$

so the tangent to the path is a Lorentz vector. It is called the 4 -velocity. It is easy to find the components of the 4-velocity in terms of the usual "3-velocity", $\mathbf{v}$,

$$
\begin{aligned}
u^{\beta} &= \frac{dx^{\beta}}{d\tau} \\
&= \frac{d}{d\tau}(ct, \mathbf{x}) \\
&= \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right) \\
&= \left(c\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{d\mathbf{x}}{dt}\right) \\
&= \left(c\gamma, \gamma\frac{d\mathbf{x}}{dt}\right) \\
&= \gamma(c, \mathbf{v})
\end{aligned}
$$

Since u^{α} is a 4-vector, its length must be something that is independent of the frame of reference of the observer. Let's compute it to check,

$$
\begin{aligned}
u^{\alpha}u_{\alpha} &= \gamma(c, \mathbf{v}) \cdot \gamma(c, -\mathbf{v}) \\
&= \gamma^{2}\left(c^{2} - \mathbf{v}^{2}\right) \\
&= \frac{c^{2} - \mathbf{v}^{2}}{1 - \frac{\mathbf{v}^{2}}{c^{2}}} \\
&= c^{2}
\end{aligned}
$$

Indeed, all observers agree on this value!

Now let m be the mass of a particle. It is natural to define the 4 -momentum,

$$
p^{\alpha} = mu^{\alpha} \tag{3}
$$

Since u^{α} is a Lorentz vector and we require p^{α} to be a Lorentz vector, the mass m must be *Lorentz invariant.* Once again, the magnitude is invariant, since $p_{\alpha}p^{\alpha} = m^{2}u_{\alpha}u^{\alpha} = m^{2}c^{2}$. Notice that if m is not Lorentz invariant, the 4-momentum is not a 4-vector. The components of p^{α} are called the (relativistic) energy and the (relativistic) 3-momentum. They are given by the familiar formulas,

$$
\begin{aligned}
p^{\alpha} &= (E/c, \mathbf{p}) \\
&= mu^{\alpha} \\
&= (m\gamma c, m\gamma\mathbf{v})
\end{aligned}
$$

Expanding the γ factor when $\mathbf{v}^{2} \ll c^{2}$,

$$
\begin{aligned}
\gamma &= \left(1 - \frac{\mathbf{v}^{2}}{c^{2}}\right)^{-1/2} \\
&= 1 + \frac{\mathbf{v}^{2}}{2c^{2}} + O\left(\frac{\mathbf{v}^{4}}{c^{4}}\right)
\end{aligned}
$$

we recover the non-relativistic expressions

$$
\begin{aligned}
E &= m\gamma c^{2} \approx mc^{2} + \frac{1}{2}m\mathbf{v}^{2} \\
\mathbf{p} &= m\gamma\mathbf{v} \approx m\mathbf{v}
\end{aligned}
$$

We will shortly see other objects with linear, homogeneous transformations under the Lorentz group. Some have multiple indices,

$$T^{\alpha\beta\ldots\mu}$$

and transform linearly on each index,

$$(T')^{\alpha\beta\ldots\mu} = \Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}\Lambda^{\mu}{}_{\nu}T^{\rho\sigma\ldots\nu} \tag{4}$$

The collection of all such objects is called the set of Lorentz tensors. More specifically, we are discussing the group of transformations that preserves the matrix $diag(-1,1,1,1)$. This group is name $O(3,1)$, meaning the pseudo-orthogonal group that preserves the 4-dimensional metric with 3 plus and 1 minus sign. In general the group of transformations preserving $diag(1,\ldots 1,-1,\ldots -1)$ with p plus signs and q plus signs is named $O(p,q)$. From the definition of $\Lambda^{\alpha}{}_{\mu}$ via $\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}$, or, more concisely

$$\eta = \Lambda^{t}\eta\Lambda$$

we see that $(\det\Lambda)^2 = 1$. If we restrict to $\det\Lambda = +1$, the corresponding group is called $SO(3,1)$, where the S stands for "special".

3 Lorentz invariant tensors

Notice that the defining property of Lorentz transformations, eq.(2), states the invariance of the metric $\eta_{\alpha\beta}$ under Lorentz transformations. This is a very special property – in general, the components of tensors are shuffled linearly by Lorentz transformations.

The Levi-Civita tensor, defined to be the unique, totally antisymmetric rank four tensor $\varepsilon_{\alpha\beta\mu\nu}$ with

$$\varepsilon_{0123} = 1$$

is the only other independent tensor which is Lorentz invariant. To see that $\varepsilon_{\alpha\beta\mu\nu}$ is invariant, we first note that it may be used to define determinants. For any matrix $M^{\alpha\beta}$, we may write

$$\begin{aligned} \det M &= \varepsilon_{\alpha\beta\mu\nu}M^{\alpha0}M^{\beta1}M^{\mu2}M^{\nu3} \\ &= \frac{1}{4!}\varepsilon_{\gamma\delta\rho\sigma}\varepsilon_{\alpha\beta\mu\nu}M^{\alpha\gamma}M^{\beta\delta}M^{\mu\rho}M^{\nu\sigma} \\ &= \frac{1}{4!}\varepsilon^{\gamma\delta\rho\sigma}\varepsilon_{\alpha\beta\mu\nu}M^{\alpha}{}_{\gamma}M^{\beta}{}_{\delta}M^{\mu}{}_{\rho}M^{\nu}{}_{\sigma} \end{aligned}$$

because the required antisymmetrizations are accomplished by the Levi-Civita tensor. An alternative way to write this is

$$(\det M)\,\varepsilon_{\gamma\delta\rho\sigma} = \varepsilon_{\alpha\beta\mu\nu}M^{\alpha}{}_{\gamma}M^{\beta}{}_{\delta}M^{\mu}{}_{\rho}M^{\nu}{}_{\sigma}$$

because the right side is totally antisymmetric on $\gamma\delta\rho\sigma$ and if we set $\gamma\delta\rho\sigma = 0123$ we get our original expression for $\det M$. Since this last expression holds for any matrix $M^{\alpha}{}_{\gamma}$, it holds for the Lorentz transformation matrix, $\Lambda^{\alpha}{}_{\gamma}$,

$$(\det\Lambda)\,\varepsilon_{\gamma\delta\rho\sigma} = \varepsilon_{\alpha\beta\mu\nu}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}$$

However, since the determinant of a (proper) Lorentz transformation is $+1$, we have the invariance of the Levi-Civita tensor,

$$\varepsilon_{\gamma\delta\rho\sigma} = \varepsilon_{\alpha\beta\mu\nu}\Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}$$

This also shows that under spatial inversion, which has $\det\Lambda = -1$, the Levi-Civita tensor changes sign. The presence of an odd number of Levi-Civita tensors in any relativistic expression therefore shows that that expression is odd under parity.

In fact, we need only know this parity argument for a single Levi-Civita tensor, because any pair of them may always be replaced by four antisymmetrized Kronecker deltas Defining $\varepsilon^{\alpha\beta\mu\nu} \equiv \eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\mu\lambda}\eta^{\nu\tau}\varepsilon_{\rho\sigma\lambda\tau} = -\varepsilon_{\alpha\beta\mu\nu}$, we have

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\gamma\delta\rho\sigma} = -\delta^{\alpha}_{[\gamma}\delta^{\beta}_{\delta}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma]} \tag{5}$$

where the square brackets around the indices indicate antisymmetrization over all 24 permutations of $\gamma\delta\rho\sigma$, with the normalization $\frac{1}{4!}$. By taking one, two, three or four contractions we obtain the following identities

$$\begin{aligned}
\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\delta\rho\sigma} &= -6\delta^{\beta}_{[\delta}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma]} && (6)\\
\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\rho\sigma} &= -2\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\right) && (7)\\
\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\mu\sigma} &= -6\delta^{\nu}_{\sigma} && (8)\\
\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\mu\nu} &= -24 && (9)
\end{aligned}$$

Similar identities hold in every dimension. In n dimensions, the Levi-Civita tensor is of rank n. For example, the Levi-Civita tensor of Euclidean 3-space is ε_{ijk}, where $\varepsilon_{123} = 1$ and all other components follow using the antisymmetry. Along with the metric, $g_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, ε_{ijk} is invariant under $SO(3)$. It is again odd under parity, and satisfies the following identities

$$\begin{aligned}
\varepsilon^{ijk}\varepsilon_{lmn} &= \delta^{i}_{[l}\delta^{j}_{m}\delta^{k}_{n]}\\
\varepsilon^{ijk}\varepsilon_{imn} &= \delta^{j}_{m}\delta^{k}_{n} - \delta^{j}_{n}\delta^{k}_{m}\\
\varepsilon^{ijk}\varepsilon_{ijn} &= 2\delta^{k}_{n}\\
\varepsilon^{ijk}\varepsilon_{ijk} &= 6
\end{aligned}$$

These identities will be useful in our discussion of the rotation group.

4 Discrete Lorentz transformations

In addition to rotations and boosts, there are two additional discrete transformations which preserve τ. Normally these are taken to be parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$). Parity is defined as spatial inversion,

$$\mathcal{P} : (t, \mathbf{x}) \to (t, -\mathbf{x}) \tag{10}$$

We do not achieve new symmetries by reflecting only two of the spatial coordinates, e.g., $(t, x, y, z) \to (t, -x, -y, z)$ because this effect is achieved by a rotation by π about the z axis. For the same reason, reflection of a single coordinate is equivalent to reflecting all three. The effect of the parity on energy and momentum follows easily. Since the 4-momentum is defined by

$$p^{\beta} = m\frac{dx^{\beta}}{d\tau}$$

and because m and τ are Lorentz invariant, we have

$$\begin{aligned}
\mathcal{P}\left(E/c, \mathbf{p}\right) &= \mathcal{P}\left(m\frac{d\left(t, \mathbf{x}\right)}{d\tau}\right)\\
&= m\frac{d}{d\tau}\mathcal{P}\left(t, \mathbf{x}\right)\\
&= m\frac{d}{d\tau}\left(t, -\mathbf{x}\right)\\
&= \left(E/c, -\mathbf{p}\right)
\end{aligned}$$

We define the second discrete Lorentz transformation, chronicity, as follows.

Define: *Chronicity,* Θ, is the reversal of the time component of 4-vectors,

$$\Theta : \left(w^0, \mathbf{w}\right) \rightarrow \left(-w^0, \mathbf{w}\right) \tag{11}$$

This is clearly a Lorentz transformation. Notice that the effect of chronicity on energy and momentum is

$$\Theta\left(E/c, \mathbf{p}\right) = \left(-E/c, \mathbf{p}\right)$$

With this definition of the symmetry, the energy-momentum is once again a proper 4-vector.

The non-relativistic limit of chronicity is exactly opposite to Newtonian time reversal, $\mathcal{T}$, which takes $t \rightarrow -t$, and therefore,

$$\begin{aligned}
\mathcal{T}\mathbf{p} &= m\frac{d\mathbf{x}}{dt} = -\mathbf{p} \\
\mathcal{T}E &= m\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt} = E
\end{aligned}$$

Notice the unexpected role played by the invariance of the proper time. By contrast with Newtonian time reversal, with the invariance of τ and the linearity of both E and $\mathbf{p}$ in τ, only the energy reverses sense. We may resolve this difference in symmetries by imagining a transformation which reverses the motion of a system. This idea plays a role in discussing the inevitable negative energy states that arise in field theory and their relation to antiparticles.

The subgroup of Lorentz transformations for which the coordinate system remains right handed is called the *proper Lorentz group*, and the subgroup of Lorentz transformations which maintains the orientation of time is called the *orthochronous Lorentz group.* The simply connected subgroup which maintains both the direction of time and the handedness of the spatial coordinates is the *proper orthochronous Lorentz group.*