Simple Harmonic Oscillator

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One of the most important problems in quantum mechanics is the simple harmonic oscillator, in part because its properties are directly applicable to field theory.

1 Hamiltonian

Writing the potential $\frac{1}{2}kx^2$ in terms of the classical frequency, $\omega = \sqrt{\frac{k}{m}}$, puts the Hamiltonian in the form

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

resulting in the Hamiltonian operator,

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{m\omega^2 \hat{X}^2}{2}$$

We make no choice of basis.

2 Raising and lowering operators

Notice that

$$\begin{aligned} \left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right) &= x^2 + \frac{p^2}{m^2\omega^2} \\ &= \frac{2}{m\omega^2}\left(\frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}\right) \end{aligned}$$

so that we may write the classical Hamiltonian as

$$H = \frac{m\omega^2}{2}\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right)$$

We can write the quantum Hamiltonian in a similar way. Choosing our normalization with a bit of foresight, we define two conjugate operators,

$$\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{i}{m\omega}\hat{P}\right) \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{i}{m\omega}\hat{P}\right) \end{aligned}$$

The operator $\hat{a}^\dagger$ is called the raising operator and $\hat{a}$ is called the lowering operator. In taking the product of these, we must be careful with ordering since $\hat{X}$ and $\hat{P}$

$$\hat{a}^\dagger \hat{a} = \frac{m\omega}{2\hbar}\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right)$$

$$
\begin{aligned}
&= \frac{m\omega}{2\hbar}\left(\hat{X}^2 + \frac{i}{m\omega}\hat{X}\hat{P} - \frac{i}{m\omega}\hat{P}\hat{X} + \frac{\hat{P}^2}{m^2\omega^2}\right) \\
&= \frac{m\omega}{2\hbar}\left(\hat{X}^2 + \frac{i}{m\omega}\left[\hat{X},\hat{P}\right] + \frac{\hat{P}^2}{m^2\omega^2}\right)
\end{aligned}
$$

Using the commutator, $\left[\hat{X},\hat{P}\right] = i\hbar\hat{1}$, this becomes

$$
\begin{aligned}
\hat{a}^\dagger\hat{a} &= \left(\frac{1}{\hbar\omega}\right)\left(\frac{1}{2}m\omega^2\right)\left(\hat{X}^2 - \frac{\hbar}{m\omega} + \frac{\hat{P}^2}{m^2\omega^2}\right) \\
&= \frac{1}{\hbar\omega}\left(\frac{1}{2}m\omega^2\hat{X}^2 - \frac{1}{2}\hbar\omega + \frac{\hat{P}^2}{2m}\right) \\
&= \frac{1}{\hbar\omega}\left(\hat{H} - \frac{1}{2}\hbar\omega\right)
\end{aligned}
$$

and therefore,

$$
\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)
$$

3 The number operator

This turns out to be a very convenient form for the Hamiltonian because $\hat{a}$ and $a^\dagger$ have very simple properties. First, their commutator is simply

$$
\begin{aligned}
\left[\hat{a},\hat{a}^\dagger\right] &= \frac{m\omega}{2\hbar}\left[\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right),\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)\right] \\
&= \frac{m\omega}{2\hbar}\left(\left[\hat{X}, -\frac{i}{m\omega}\hat{P}\right] + \left[\frac{i}{m\omega}\hat{P},\hat{X}\right]\right) \\
&= -\frac{2i}{m\omega}\frac{m\omega}{2\hbar}\left[\hat{X},\hat{P}\right] \\
&= -\frac{i}{\hbar}i\hbar \\
&= 1
\end{aligned}
$$

Consider one further set of commutation relations. Defining $\hat{N} \equiv \hat{a}^\dagger\hat{a} = \hat{N}^\dagger$, called the number operator, we have

$$
\begin{aligned}
\left[\hat{N},\hat{a}\right] &= \left[\hat{a}^\dagger\hat{a},\hat{a}\right] \\
&= \hat{a}^\dagger\left[\hat{a},\hat{a}\right] + \left[\hat{a}^\dagger,\hat{a}\right]\hat{a} \\
&= -\hat{a}
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\hat{N},\hat{a}^\dagger\right] &= \left[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\right] \\
&= \hat{a}\left[\hat{a}^\dagger,\hat{a}^\dagger\right] + \hat{a}^\dagger\left[\hat{a},\hat{a}^\dagger\right] \\
&= \hat{a}^\dagger
\end{aligned}
$$

Notice that $\hat{N}$ is Hermitian, hence observable, and that $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$.

4 Energy eigenkets

4.1 Positivity of the energy

Consider an arbitrary expectation value of the Hamiltonian,

$$\begin{aligned}
\langle\psi|\,\hat{H}\,|\psi\rangle &= \langle\psi|\,\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)|\psi\rangle \\
&= \hbar\omega\left(\langle\psi|\,\hat{a}^{\dagger}\hat{a}\,|\psi\rangle+\frac{1}{2}\,\langle\psi\,|\psi\rangle\right)
\end{aligned}$$

Since $\langle\psi\,|\psi\rangle > 0$ for any state

$$\begin{aligned}
\langle\psi|\,\hat{H}\,|\psi\rangle &= \hbar\omega\left(\langle\psi|\,\hat{a}^{\dagger}\hat{a}\,|\psi\rangle+\frac{1}{2}\,\langle\psi\,|\psi\rangle\right) \\
&> \hbar\omega\,\langle\psi|\,\hat{a}^{\dagger}\hat{a}\,|\psi\rangle
\end{aligned}$$

and if we define $|\beta\rangle \equiv \hat{a}\,|\psi\rangle$ we see that the remaining term is also positive definite,

$$\langle\psi|\,\hat{a}^{\dagger}\hat{a}\,|\psi\rangle = \langle\beta\,|\beta\rangle > 0$$

This means that all expectation values of the Hamiltonian are positive definite, and in particular, all energies are positive, since for any normalized energy eigenket,

$$\langle E|\,\hat{H}\,|E\rangle = E > 0$$

4.2 The lowest energy

Now suppose $|E\rangle$ is any normalized energy eigenket. Then consider the new ket found by acting on this state with the lowering operator, $\hat{a}$. Applying the Hamiltonian operator to $\hat{a}\,|E\rangle$,

$$\begin{aligned}
\hat{H}\left(\hat{a}\,|E\rangle\right) &= \hbar\omega\left(\hat{N}+\frac{1}{2}\right)\left(\hat{a}\,|E\rangle\right) \\
&= \hbar\omega\hat{N}\hat{a}\,|E\rangle+\frac{1}{2}\hbar\omega\hat{a}\,|E\rangle \\
&= \hbar\omega\left(\left[\hat{N},\hat{a}\right]+\hat{a}\hat{N}\right)|E\rangle+\hat{a}\frac{1}{2}\hbar\omega\left(|E\rangle\right) \\
&= \hbar\omega\left(-\hat{a}+\hat{a}\hat{N}\right)|E\rangle+\hat{a}\frac{1}{2}\hbar\omega\,|E\rangle \\
&= -\hbar\omega\hat{a}\,|E\rangle+\hat{a}\hbar\omega\left(\hat{N}+\frac{1}{2}\right)|E\rangle \\
&= -\hbar\omega\hat{a}\,|E\rangle+\hat{a}\hat{H}\,|E\rangle \\
&= -\hbar\omega\hat{a}\,|E\rangle+\hat{a}E\,|E\rangle \\
&= \left(E-\hbar\omega\right)\left(\hat{a}\,|E\rangle\right)
\end{aligned}$$

This means that $\hat{a}\,|E\rangle$ is also an energy eigenket, with energy $E-\hbar\omega$. Since $\hat{a}\,|E\rangle$ is an energy eigenket, we may repeat this procedure to show that $\hat{a}^2\,|E\rangle$ is an energy eigenket with energy $E-2\hbar\omega$. Continuing in this way, we find that $\hat{a}^k\,|E\rangle$ will have energy $E-k\hbar\omega$. This process cannot continue indefinitely, because the energy must remain positive. Let k be the largest integer for which $E-k\hbar\omega$ is positive,

$$\hat{H}\hat{a}^k\,|E\rangle = \left(E-k\hbar\omega\right)\hat{a}^k\,|E\rangle$$

with corresponding state $\hat{a}^k|E\rangle$. Then applying the lowering operator one more time cannot give a new state. The only other possibility is zero. Rename the lowest energy state $|0\rangle = A_0\hat{a}^k|E\rangle$, where we choose A_0 so that $|0\rangle$ is normalized. We then must have

$$\hat{a}|0\rangle = 0$$

and therefore,

$$\begin{aligned} \hat{H}|0\rangle &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle \\ &= \frac{1}{2}\hbar\omega|0\rangle \end{aligned}$$

This is the lowest energy state of the oscillator.

To see that it is unique, suppose we had chosen a different energy eigenket, $|E'\rangle$, to start with. Then we would find a new ground state, $|0'\rangle$, also satisfying $\hat{a}|0'\rangle = 0$. However, as we show in the Section 5, the condition $\hat{a}|0\rangle = 0$ in a coordinate basis leads to a differential equation with a unique solution for the ground state wave function. Thus, there is only one state satisfying $\hat{a}|0\rangle = 0$.

4.3 The complete spectrum

Now that we have the ground state, we reverse the process, acting instead with the raising operator. Acting on any energy eigenket, we have

$$\begin{aligned} \hat{H}\left(\hat{a}^\dagger|E\rangle\right) &= \hbar\omega\left(\hat{N} + \frac{1}{2}\right)\left(\hat{a}^\dagger|E\rangle\right) \\ &= \hbar\omega\hat{N}\hat{a}^\dagger|E\rangle + \frac{1}{2}\hbar\omega\hat{a}^\dagger|E\rangle \\ &= \hbar\omega\left(\left[\hat{N},\hat{a}^\dagger\right] + \hat{a}^\dagger\hat{N}\right)|E\rangle + \hat{a}^\dagger\frac{1}{2}\hbar\omega|E\rangle \\ &= \hbar\omega\hat{a}^\dagger|E\rangle + \hat{a}^\dagger\left(\hbar\omega\hat{N} + \frac{1}{2}\hbar\omega\right)|E\rangle \\ &= \hbar\omega\hat{a}^\dagger|E\rangle + \hat{a}^\dagger\hat{H}|E\rangle \\ &= \hbar\omega\hat{a}^\dagger|E\rangle + \hat{a}^\dagger E|E\rangle \\ &= (E + \hbar\omega)\left(\hat{a}^\dagger|E\rangle\right) \end{aligned}$$

Therefore, beginning with this lowest state, we have

$$\begin{aligned} \hat{H}\left(\hat{a}^\dagger|0\rangle\right) &= \left(\hbar\omega + \frac{1}{2}\hbar\omega\right)\left(\hat{a}^\dagger|0\rangle\right) \\ &= \frac{3}{2}\hbar\omega\left(\hat{a}^\dagger|E\rangle\right) \end{aligned}$$

and we define the normalized state to be

$$|1\rangle = A_1\hat{a}^\dagger|0\rangle$$

There is nothing to prevent us continuing this procedure indefintely. Continuing, we have states

$$|n\rangle = A_n\left(\hat{a}^\dagger\right)^n|0\rangle$$

satisfying

$$\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle$$

This gives the complete set of energy eigenkets.

4.4 Normalization

We have defined the lowest ket, $|0\rangle$, to be normalized. For the next level, we require

$$\begin{aligned}
1 &= \langle 1 | 1 \rangle \\
&= |A_1|^2 \langle 0| \hat{a}\hat{a}^\dagger |0\rangle \\
&= |A_1|^2 \langle 0| \left(\left[\hat{a}, \hat{a}^\dagger \right] + \hat{a}^\dagger \hat{a} \right) |0\rangle \\
&= |A_1|^2 \langle 0| \left(1 + \hat{a}^\dagger \hat{a} \right) |0\rangle \\
&= |A_1|^2 \langle 0 | 0 \rangle
\end{aligned}$$

so that choosing the phase so that A_1 is real, we have $A_1 = 1$.

Now, consider the expectation of $\hat{N}$ in the n^{th} state. We see from the energy that the eigenvalues of the number operator are integers, n, so that for the normalized state $|n\rangle$,

$$\begin{aligned}
1 &= \langle n | n \rangle \\
&= |A_n|^2 \langle 0| \hat{a}^n \left(\hat{a}^\dagger \right)^n |0\rangle \\
&= |A_n|^2 \langle 0| \hat{a}^{n-1} \hat{a}\hat{a}^\dagger \left(\hat{a}^\dagger \right)^{n-1} |0\rangle \\
&= |A_n|^2 \langle 0| \hat{a}^{n-1} \left(\hat{a}^\dagger \hat{a} + \left[\hat{a}, \hat{a}^\dagger \right] \right) \left(\hat{a}^\dagger \right)^{n-1} |0\rangle \\
&= |A_n|^2 \langle 0| \hat{a}^{n-1} \left(\hat{N} + 1 \right) \left(\hat{a}^\dagger \right)^{n-1} |0\rangle \\
&= |A_n|^2 \left(\langle n-1| \frac{1}{A_{n-1}^*} \right) \left(\hat{N} + 1 \right) \left(\frac{1}{A_{n-1}} |n-1\rangle \right) \\
&= \frac{|A_n|^2}{|A_{n-1}|^2} \langle n-1| \left(\hat{N} + 1 \right) |n-1\rangle \\
&= \frac{|A_n|^2}{|A_{n-1}|^2} (n - 1 + 1)
\end{aligned}$$

Therefore, $|A_{n-1}|^2 = n |A_n|^2$, so iterating,

$$\begin{aligned}
|A_n|^2 &= \frac{1}{n} |A_{n-1}|^2 \\
&= \frac{1}{n(n-1)} |A_{n-2}|^2 \\
&\vdots \\
&= \frac{1}{n!} |A_1|^2
\end{aligned}$$

so that, choosing all of the coefficients real, we have

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger \right)^n |0\rangle$$

5 Wave function

Now consider the wave function, $\psi_n(x)$, for the eigenstates. For the lowest state, we know that

$$\hat{a} |0\rangle = 0$$

so in a coordinate basis, we compute

$$
\begin{aligned}
0 &= \langle x|\,\hat{a}\,|0\rangle \\
&= \sqrt{\frac{m\omega}{2\hbar}}\,\langle x|\left(\hat{X}+\frac{i}{m\omega}\hat{P}\right)|0\rangle \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(\langle x|\,\hat{X}\,|0\rangle+\frac{i}{m\omega}\hat{P}\,|0\rangle\right) \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(x\,\langle x\,|0\rangle+\frac{i}{m\omega}\,\langle x|\,\hat{P}\,|0\rangle\right)
\end{aligned}
$$

where, inserting an identity,

$$
\begin{aligned}
\langle x|\,\hat{P}\,|0\rangle &= \int dx'\,\langle x|\,\hat{P}\,|x'\rangle\,\langle x'\,|0\rangle \\
&= \int dx'\left(i\hbar\frac{\partial}{\partial x'}\delta^3\left(x-x'\right)\right)\langle x'\,|0\rangle \\
&= -i\hbar\int dx'\delta^3\left(x-x'\right)\frac{\partial}{\partial x'}\,\langle x'\,|0\rangle \\
&= -i\hbar\frac{d}{dx}\,\langle x\,|0\rangle
\end{aligned}
$$

Therefore, setting $\psi_0\left(x\right)=\langle x\,|0\rangle$ and substituting,

$$
\begin{aligned}
0 &= x\,\langle x\,|0\rangle+\frac{i}{m\omega}\,\langle x|\,\hat{P}\,|0\rangle \\
&= x\psi_0\left(x\right)+\frac{i}{m\omega}\left(-i\hbar\frac{d}{dx}\psi_0\left(x\right)\right) \\
&= x\psi_0\left(x\right)+\frac{\hbar}{m\omega}\frac{d}{dx}\psi_0\left(x\right) \\
\frac{d}{dx}\psi_0\left(x\right) &= -\frac{m\omega x}{\hbar}\psi_0\left(x\right) \\
\psi_0\left(x\right) &= Ae^{-\frac{m\omega x^2}{2\hbar}}
\end{aligned}
$$

so the wave function of the ground state is a Gaussian.

To find the wave functions of the higher energy states, consider

$$
\begin{aligned}
\psi_n\left(x\right) &= \langle x\,|n\rangle \\
&= \langle x|\,\frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle \\
&= \frac{1}{\sqrt{n}}\,\langle x|\,\hat{a}^\dagger\frac{1}{\sqrt{(n-1)!}}\left(\hat{a}^\dagger\right)^{n-1}|0\rangle \\
&= \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\,\langle x|\left(\hat{X}-\frac{i}{m\omega}\hat{P}\right)|n-1\rangle \\
&= \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\left(x\,\langle x\,|n-1\rangle-\frac{i}{m\omega}\int dx'\,\langle x|\,\hat{P}\,|x'\rangle\,\langle x'\,|n-1\rangle\right) \\
&= \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\left(x\psi_{n-1}\left(x\right)-\frac{i}{m\omega}\int dx'\left(i\hbar\frac{\partial}{\partial x'}\delta^3\left(x-x'\right)\right)\psi_{n-1}\left(x'\right)\right) \\
&= \frac{1}{\sqrt{n}}\sqrt{\frac{m\omega}{2\hbar}}\left(x\psi_{n-1}\left(x\right)-\frac{\hbar}{m\omega}\int dx'\delta^3\left(x-x'\right)\frac{\partial}{\partial x'}\psi_{n-1}\left(x'\right)\right) \\
&= \sqrt{\frac{m\omega}{2n\hbar}}\left(x-\frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_{n-1}\left(x\right)
\end{aligned}
$$

Therefore, we can find all states by iterating this operator,

$$\psi_n\left(x'\right) = \sqrt{\frac{m\omega}{2n!\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n \psi_0\left(x\right)$$

The result is a series of polynomials, the Hermite polynomials, times the Gaussian factor.

Exercise: Find $\psi_1(x)$ and $\psi_2(x)$.

6 Time evolution of a mixed state of the oscillator

Consider the time evolution of a state of the harmonic oscillator given by the most general superposition of the lowest two eigenstates

$$|\psi\rangle = \cos\theta\,|0\rangle + e^{i\varphi}\sin\theta\,|1\rangle$$

The time evolution is given by

$$\begin{aligned}
|\psi, t\rangle &= \mathcal{U}(t)\,|\psi\rangle \\
&= e^{-\frac{i}{\hbar}\hat{H}t}\,|\psi\rangle \\
&= \cos\theta e^{-\frac{i}{\hbar}\hat{H}t}\,|0\rangle + e^{i\varphi}\sin\theta e^{-\frac{i}{\hbar}\hat{H}t}\,|1\rangle \\
&= \cos\theta e^{-\frac{i}{\hbar}E_0 t}\,|0\rangle + e^{i\varphi}\sin\theta e^{-\frac{i}{\hbar}E_1 t}\,|1\rangle \\
&= \cos\theta e^{-\frac{i}{2}\omega t}\,|0\rangle + e^{i\varphi}\sin\theta e^{-\frac{3}{2}i\omega t}\,|1\rangle \\
&= e^{-\frac{i}{2}\omega t}\left(\cos\theta\,|0\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}\,|1\rangle\right)
\end{aligned}$$

Now look at the time dependence of the expectation value of the position operator, which we write in terms of raising and lowering operators as $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$:

$$\begin{aligned}
\langle\psi, t|\,\hat{X}\,|\psi, t\rangle &= \left(\cos\theta\,\langle 0| + e^{-i\varphi}\sin\theta e^{i\omega t}\,\langle 1|\right) e^{\frac{i}{2}\omega t}\hat{X}e^{-\frac{i}{2}\omega t}\left(\cos\theta\,|0\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}\,|1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\cos\theta\,\langle 0| + e^{-i\varphi}\sin\theta e^{i\omega t}\,\langle 1|\right)\left(\hat{a} + \hat{a}^\dagger\right)\left(\cos\theta\,|0\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}\,|1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\cos\theta\,\langle 0| + e^{-i\varphi}\sin\theta e^{i\omega t}\,\langle 1|\right)\left(\cos\theta\,|1\rangle + e^{i\varphi}\sin\theta e^{-i\omega t}\left(|0\rangle + \sqrt{2}\,|2\rangle\right)\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\cos\theta\sin\theta e^{-i(\omega t - \varphi)} + \sin\theta\cos\theta e^{i(\omega t - \varphi)}\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\sin 2\theta\cos\left(\omega t - \varphi\right)
\end{aligned}$$

where we have used $\hat{a}\,|0\rangle = 0, \hat{a}\,|1\rangle = |0\rangle\,, \hat{a}^\dagger\,|0\rangle = |1\rangle$ and $\hat{a}^\dagger\,|1\rangle = \sqrt{2}\,|2\rangle$. We see that the expected position oscillates back and forth between $\pm\sqrt{\frac{\hbar}{2m\omega}}\sin 2\theta$ with frequency ω. Superpositions involving higher excited states will bring in harmonics, $n\omega$, and will then allow for varied traveling waveforms.

7 Coherent states

We define a coherent state of the harmonic oscillator to be an eigenstate of the lowering operator,

$$\hat{a}\,|\lambda\rangle = \lambda\,|\lambda\rangle$$

To find this state, let

$$|\lambda\rangle = \sum_n c_n |n\rangle$$

then require

$$\begin{aligned}
\hat{a} \sum_n c_n |n\rangle &= \lambda \sum_n c_n |n\rangle \\
\sum_n \frac{c_n}{\sqrt{n!}} \hat{a} \left(\hat{a}^\dagger\right)^n |0\rangle &= \lambda \sum_n c_n |n\rangle \\
\sum_n \frac{c_n}{\sqrt{n!}} \left(\left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] + \left(\hat{a}^\dagger\right)^n \hat{a}\right) |0\rangle &= \lambda \sum_n c_n |n\rangle \\
\sum_n \frac{c_n}{\sqrt{n!}} \left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] |0\rangle &= \lambda \sum_n c_n |n\rangle
\end{aligned}$$

7.1 Computing the commutators

Now find the commutators $\hat{A}_n \equiv \left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right]$. To begin, look at the first few. Since $\hat{A}_1 = \left[\hat{a}, \hat{a}^\dagger\right] = 1$,

$$\begin{aligned}
\hat{A}_1 &= \left[\hat{a}, \hat{a}^\dagger\right] = 1 \\
\hat{A}_2 &= \left[\hat{a}, \left(\hat{a}^\dagger\right)^2\right] \\
&= \hat{a}^\dagger \left[\hat{a}, \hat{a}^\dagger\right] + \left[\hat{a}, \hat{a}^\dagger\right] \hat{a}^\dagger \\
&= 2\hat{a}^\dagger \\
\hat{A}_3 &= \left[\hat{a}, \left(\hat{a}^\dagger\right)^3\right] \\
&= \hat{a}^\dagger \left[\hat{a}, \left(\hat{a}^\dagger\right)^2\right] + \left[\hat{a}, \hat{a}^\dagger\right] \left(\hat{a}^\dagger\right)^2 \\
&= 2\left(\hat{a}^\dagger\right)^2 + \left(\hat{a}^\dagger\right)^2 \\
&= 3\left(\hat{a}^\dagger\right)^2
\end{aligned}$$

This suggests that $\hat{A}_n = n\left(\hat{a}^\dagger\right)^{n-1}$. We prove it by induction. First, the relation is true for $n = 1$. Now, assume it holds for $n - 1$, and try to prove that it must hold for n. If it holds for $n - 1$, then

$$\hat{A}_{n-1} = (n-1)\left(\hat{a}^\dagger\right)^{n-2}$$

and we compute $\hat{A}_n$:

$$\begin{aligned}
\hat{A}_n &\equiv \left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] \\
&= \hat{a}^\dagger \left[\hat{a}, \left(\hat{a}^\dagger\right)^{n-1}\right] + \left[\hat{a}, \hat{a}^\dagger\right] \left(\hat{a}^\dagger\right)^{n-1} \\
&= \hat{a}^\dagger \hat{A}_{n-1} + \left(\hat{a}^\dagger\right)^{n-1} \\
&= \hat{a}^\dagger (n-1)\left(\hat{a}^\dagger\right)^{n-2} + \left(\hat{a}^\dagger\right)^{n-1} \\
&= n\left(\hat{a}^\dagger\right)^{n-1}
\end{aligned}$$

which is the anticipated result for n. Since the supposition is true for $n = 1$, and is true for n whenever it holds for $n - 1$, it holds for all integers.

7.2 A recursion relation for coherent states

Now return to our condition for coherence,

$$\sum_n \frac{c_n}{\sqrt{n!}} \left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] |0\rangle = \lambda \sum_n c_n |n\rangle$$

Substituting for the commutators, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} n \left(\hat{a}^\dagger\right)^{n-1} |0\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle \\
\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n!}} n \left(\hat{a}^\dagger\right)^{n-1} |0\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle \\
\sum_{n=1}^{\infty} \frac{c_n}{\sqrt{n!}} n \sqrt{(n-1)!}\, |n-1\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle \\
\sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle &= \lambda \sum_{n=0}^{\infty} c_n |n\rangle
\end{aligned}$$

Now rewrite the sum on the left, letting $n-1 \to n$,

$$\sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}\, |n\rangle$$

The sums now match provided

$$c_{n+1} \sqrt{n+1} = \lambda c_n$$

Iterating this recursion relationship, we find

$$c_n = \frac{\lambda^n}{\sqrt{n!}}$$

for all n. The coherent state is therefore given by

$$\begin{aligned}
|\lambda\rangle &= \sum_n \frac{\lambda^n}{\sqrt{n!}} |n\rangle \\
&= \sum_n \frac{\lambda^n}{n!} \left(\hat{a}^\dagger\right)^n |0\rangle \\
&= e^{\lambda \hat{a}^\dagger} |0\rangle
\end{aligned}$$

7.3 Time dependence

The time dependence is given by

$$\begin{aligned}
|\lambda, t\rangle &= \hat{U}(t, t_0) |\lambda, t_0\rangle \\
& \quad e^{-\frac{i}{\hbar}\hat{H}t} \sum_n \frac{\lambda^n}{\sqrt{n!}} |n\rangle \\
&= \sum_n \frac{\lambda^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}\hat{H}t} |n\rangle \\
&= \sum_n \frac{\lambda^n}{\sqrt{n!}} e^{-i\left(n+\frac{1}{2}\right)\omega t} |n\rangle
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\frac{i\omega t}{2}} \sum_n \frac{\lambda^n}{\sqrt{n!}} e^{-in\omega t} |n\rangle \\
&= e^{-\frac{i\omega t}{2}} \sum_n \frac{\left(\lambda e^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle \\
&= e^{-\frac{i\omega t}{2}} \left|\lambda e^{-i\omega t}, t_0\right\rangle
\end{aligned}
$$

so the complex parameter λ is just replaced by $\lambda e^{-i\omega t}$ in the original state.