Field theory

January 27, 2016

Until the successes of string theory, the idea of a field as a dynamical variable depending continuously on position and time has been the most successful model for matter, and still forms the basis for general relativity and the standard model of particle physics. The most familiar examples of fields are the electric and magnetic fields, and the Schrödinger wave function, but there are others types of fields as well. Probably the simplest field theory is the one that arises from Newtonian gravity. We explore the Newtonian case first, then look at aspects of electromagnetism.

1 Newtonian gravity as a field theory

1.1 From Newton's law to a field theory

Let the origin of our coordinates be placed at a mass M, with a second mass, m, at position $\mathbf{x}$. Then Newton's law of universal gravitation gives the gravitational force between them as

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{x}$ and the minus sign shows that the force is attractive. From this, we define the *gravitational field* as the gravitational acceleration, or the force per unit mass produced by M,

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

If M is at position $\mathbf{x}'$ rather than the origin, we may write $r = |\mathbf{x} - \mathbf{x}'|$ and $\hat{\mathbf{r}} = \frac{1}{r}(\mathbf{x} - \mathbf{x}')$, so that

$$\mathbf{g}(\mathbf{x}) = -\frac{GM(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

To generalize this to a field theory, we introduce a continuously varying source. Suppose matter is spread non-uniformly throughout some region, V. In any infinitesimal region d^3x' in V about a point $\mathbf{x}'$, let the amount of mass be given by

$$dm(\mathbf{x}') = \rho(\mathbf{x}')\, d^3x'$$

The function $\rho(\mathbf{x}')$ is the mass density. The contribution to the gravitational field at $\mathbf{x}$ due to dm is therefore

$$d\mathbf{g}(\mathbf{x}) = -\frac{Gdm(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$

and we may combine all such contributions by integrating,

$$\mathbf{g}(\mathbf{x}) = -G\int\limits_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}(\mathbf{x} - \mathbf{x}')\, d^3x'$$

This gives the gravitational field at any point to to an arbitrary mass density.

Because we may write the $\mathbf{x}$-dependent factor as a gradient,

$$\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

there exists a potential for the field,

$$\begin{aligned}
\Phi(\mathbf{x}) &= G\int\limits_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \\
\mathbf{g}(\mathbf{x}) &= -\boldsymbol{\nabla}\Phi(\mathbf{x})
\end{aligned}$$

and from the Laplacian of $\frac{1}{r}$,

$$\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$$

we see that the Laplacian of the potential,

$$\begin{aligned}
\nabla^2\Phi(\mathbf{x}) &= G\int\limits_V \nabla^2 \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' \\
&= -4\pi G\int\limits_V \rho(\mathbf{x}')\,\delta^3(\mathbf{x}-\mathbf{x}')\,d^3x' \\
&= -4\pi G\rho(\mathbf{x})
\end{aligned}$$

satisfies the Poisson equation,

$$\nabla^2\Phi(\mathbf{x}) = -4\pi G\rho(\mathbf{x})$$

This is now a field theory of Newtonian gravity.

1.2 An action functional for the gravitational field

In order to quantize a field theory, we generally require an action functional. The action may be any functional whose variation gives the field equation we desire, but it is instructive to work from the particle action to get corresponding result.

Suppose we have a fixed source mass M which produces a gravitational field $\mathbf{g}(\mathbf{x})$. Then the potential energy of a mass m at $\mathbf{x}$ is just m times the gravitational potential:

$$\begin{aligned}
V(\mathbf{x}) &= -\int\limits_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F}\cdot d\mathbf{x} \\
&= -m\int\limits_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{g}\cdot d\mathbf{x} \\
&= m\int\limits_{\mathbf{x}_0}^{\mathbf{x}} \boldsymbol{\nabla}\Phi(\mathbf{x})\cdot d\mathbf{x} \\
&= m\Phi(\mathbf{x})
\end{aligned}$$

Consider the energy required to assemble a mass distribution. After placing the first mass, m_1, to position $\mathbf{x}_1$ we bring a second, m_2, to position $\mathbf{x}_2$ and so on. The energy of the first pair is

$$V_2 = -\frac{Gm_1m_2}{|\mathbf{x}_1-\mathbf{x}_2|}$$

and adding a third we get contributions from both m_1 and m_2,

$$\begin{aligned}
V_3 &= -\frac{Gm_1m_2}{|\mathbf{x}_1-\mathbf{x}_2|}-\frac{Gm_1m_3}{|\mathbf{x}_1-\mathbf{x}_3|}-\frac{Gm_2m_3}{|\mathbf{x}_2-\mathbf{x}_3|}\\
&= -G\sum_{i=1}^{3}\sum_{j>i}^{3}\frac{m_im_j}{|\mathbf{x}_i-\mathbf{x}_j|}\\
&= -\frac{1}{2}G\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{m_im_j}{|\mathbf{x}_i-\mathbf{x}_j|}
\end{aligned}$$

This generalizes to $V_N=-\frac{1}{2}G\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{m_im_j}{|\mathbf{x}_i-\mathbf{x}_j|}$. Now let the masses be replaced by mass densities and the sums by integrals,

$$\begin{aligned}
V &= -\frac{1}{2}G\int d^3x\int d^3y\frac{\rho(\mathbf{x})\,\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\\
&= -\frac{1}{8\pi G}G\int d^3x\int d^3y\frac{\nabla^2\Phi(\mathbf{x})\,\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\\
&= -\frac{1}{8\pi G}G\int d^3x\int d^3y\frac{\boldsymbol{\nabla}_x\cdot\boldsymbol{\nabla}_x\Phi(\mathbf{x})\,\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\\
&= -\frac{1}{8\pi G}G\int d^3x\int d^3y\rho(\mathbf{y})\left[\boldsymbol{\nabla}_x\cdot\left(\frac{\boldsymbol{\nabla}_x\Phi(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}\right)-\boldsymbol{\nabla}_x\Phi(\mathbf{x})\cdot\boldsymbol{\nabla}_x\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\right]
\end{aligned}$$

The divergence term integrates to the boundary at infinity and vanishes, and we integrate by parts again,

$$\begin{aligned}
V &= -\frac{1}{8\pi}\int d^3x\int d^3y\rho(\mathbf{y})\left[-\boldsymbol{\nabla}_x\Phi(\mathbf{x})\cdot\boldsymbol{\nabla}_x\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\right]\\
&= -\frac{1}{8\pi}\int d^3x\int d^3y\rho(\mathbf{y})\left[-\boldsymbol{\nabla}_x\cdot\left(\Phi(\mathbf{x})\,\boldsymbol{\nabla}_x\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\right)+\Phi(\mathbf{x})\,\nabla_x^2\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\right]\\
&= -\frac{1}{8\pi}\int d^3x\int d^3y\rho(\mathbf{y})\,\Phi(\mathbf{x})\,\nabla_x^2\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)
\end{aligned}$$

The Laplacian gives a Dirac delta function, so

$$\begin{aligned}
V &= 4\pi\frac{1}{8\pi}\int d^3x\int d^3y\rho(\mathbf{y})\,\Phi(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{y})\\
&= \frac{1}{2}\int d^3x\rho(\mathbf{x})\,\Phi(\mathbf{x})
\end{aligned}$$

Using the field equation, we may express this in terms of the gravitational field only. Replacing ρ,

$$V = \frac{1}{2}\int d^3x\,\Phi\nabla^2\Phi$$

This takes a more symmetrical form if we integrate by parts. Discarding the surface term,

$$\begin{aligned}
V &= \frac{1}{2}\int d^3x\,\Phi\nabla^2\Phi\\
&= -\frac{1}{8\pi G}\int d^3x\,\boldsymbol{\nabla}\Phi(\mathbf{x})\cdot\boldsymbol{\nabla}\Phi(\mathbf{x})\\
&= -\frac{1}{8\pi G}\int d^3x\,\mathbf{g}^2
\end{aligned}$$

and we identify the field energy density

$$w = -\frac{\mathbf{g}^2}{8\pi G}$$

This energy may be thought of as existing even in the free space between matter, as energy of the field. *In addition*, if we move a mass m into the region, it will have potential energy $m\Phi$. Therefore, the potential energy of the matter together with the potential energy of the field is given by

$$V = \int d^3x \left[-\frac{1}{8\pi G}\boldsymbol{\nabla}\Phi\left(\mathbf{x}\right)\cdot\boldsymbol{\nabla}\Phi\left(\mathbf{x}\right)+\rho\left(\mathbf{x}\right)\Phi\left(\mathbf{x}\right)\right]$$

Since Newtonian gravity is not dynamical, there is no kinetic energy associated with Φ, and the Lagrangian is just $L = -V$. The action is therefore

$$S = \int dt \int d^3x \left[\frac{1}{8\pi G}\boldsymbol{\nabla}\Phi\left(\mathbf{x}\right)\cdot\boldsymbol{\nabla}\Phi\left(\mathbf{x}\right)-\rho\left(\mathbf{x}\right)\Phi\left(\mathbf{x}\right)\right]$$

We find the field equation by varying the *field*. It is always the dynamical variable that is varied, whether it describes a curve, $\mathbf{x}(t)$, or a field $\Phi(\mathbf{x}, t)$. In the present case,

$$\begin{aligned}
0 &= \delta S \\
&= \delta\int dt \int d^3x \left[\frac{1}{8\pi G}\boldsymbol{\nabla}\Phi\left(\mathbf{x}\right)\cdot\boldsymbol{\nabla}\Phi\left(\mathbf{x}\right)-\rho\left(\mathbf{x}\right)\Phi\left(\mathbf{x}\right)\right] \\
&= \int dt \int d^3x \left[\frac{1}{8\pi G}2\boldsymbol{\nabla}\delta\Phi\cdot\boldsymbol{\nabla}\Phi-\rho\left(\mathbf{x}\right)\delta\Phi\left(\mathbf{x}\right)\right] \\
&= \int dt \int d^3x \left[\frac{1}{4\pi G}\left[\boldsymbol{\nabla}\cdot\left(\delta\Phi\boldsymbol{\nabla}\Phi\right)-\delta\Phi\nabla^2\Phi\right]-\rho\delta\Phi\right] \\
&= \frac{1}{4\pi G}\int dt \int d^3x\, \boldsymbol{\nabla}\cdot\left(\delta\Phi\boldsymbol{\nabla}\Phi\right) - \int dt \int d^3x \left[\frac{1}{4\pi G}\nabla^2\Phi+\rho\right]\delta\Phi \\
&= \frac{1}{4\pi G}\int dt \int d^2x\, \hat{\mathbf{n}}\cdot\left(\delta\Phi\boldsymbol{\nabla}\Phi\right) - \int dt \int d^3x \left[\frac{1}{4\pi G}\nabla^2\Phi+\rho\right]\delta\Phi \\
&= -\int dt \int d^3x \left[\frac{1}{4\pi G}\nabla^2\Phi+\rho\right]\delta\Phi
\end{aligned}$$

Since $\delta\Phi$ is arbitrary the term in brackets must vanish, and we recover the field equation,

$$\nabla^2\Phi = -4\pi G\rho$$

2 Dynamical gravity

One, but not the only, failing of Newtonian gravity is its instantaneous propagation. It is not difficult to write a theory of gravity in which gravitational waves propagate with speed $\mathbf{v}$. Taking inspiration from the equation

$$-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial x^2} = 0$$

for waves on a string, and generalizing the spatial second derivative $\frac{\partial u}{\partial x^2}$ to the Laplacian gives a 3-dimensional wave equation,

$$-\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}+\nabla^2 u = 0$$

This has immediate wave solutions,

$$u\left(\mathbf{x},t\right) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

Newtonian gravity will also have wave solutions if we generalize it to

$$-\frac{1}{v^2}\frac{\partial^2\Phi}{\partial t^2}+\nabla^2\Phi=-4\pi G\rho$$

or, if we wish it to be consistent with special relativity,

$$-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}+\nabla^2\Phi=-4\pi G\rho$$

Any static solution agrees with the usual Newtonian result, but now we will show that the gravitational influence propagates at the speed of light. To do this, we compute the Green function for the field equation.

2.1 The Helmholz equation

Our propagating gravity equation has the form

$$\Box\Phi=-4\pi G_N\rho\left(\mathbf{x},t\right)$$

where the d'Alembertian, $\Box$, is the wave operator, given by

$$\Box=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2$$

and ∇^2 is the usual 3-dimensional Laplacian. We write G_N for Newton's graviatational constant so as not to confuse it with the Green function. We wish to solve this equation in the absence of boundaries. This means the boundary conditions reduce to regularity at the origin and vanishing at infinity.

There are three steps to solving. First we solve the homogeneous equation,

$$\Box\phi=0$$

Then, we use homogeneous solutions together with a boundary condition to solve for the empty space Green's function,

$$\Box G\left(\mathbf{x},t;\mathbf{x}',t'\right)=-4\pi G_N\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right)$$

Finally, we construct the full solution for Φ by integrating the source with the Green's function,

$$\Phi\left(\mathbf{x},t\right)=\int G\left(\mathbf{x},t;\mathbf{x}',t'\right)\rho\left(\mathbf{x}',t'\right)d^3x'dt'$$

It is immediate that applying the d'Alembertion to Φ, interchanging primed integrations with the unprimed derivatives of the d'Alembertion, then integrating over the resulting delta functions reproduces the equation we wish to solve.

We vary this procedure in one significant way, by performing a Fourier transformation on the time coordinate. This is a practical consideration, since it immediately puts things in terms of electromagnetic waves, a topic we will consider in detail. The Fourier transformation is the replacement in the original equation of Φ and ρ by integrals over frequency,

$$\begin{aligned}\Phi\left(\mathbf{x},t\right)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\\ \rho\left(\mathbf{x},t\right)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\end{aligned}$$

Substituting these into the wave equation we have

$$\begin{aligned}
\Box\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\Phi\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\right) &= -4\pi G_N\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\rho\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\right)\\
\int\limits_{-\infty}^{\infty}\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\left(\Phi\left(\mathbf{x},\omega\right)e^{-i\omega t}\right)d\omega &= -4\pi G_N\int\limits_{-\infty}^{\infty}\rho\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\\
\int\limits_{-\infty}^{\infty}\left(-\Phi\left(\mathbf{x},\omega\right)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}e^{-i\omega t}+e^{-i\omega t}\nabla^2\Phi\left(\mathbf{x},\omega\right)\right)d\omega &= -4\pi G_N\int\limits_{-\infty}^{\infty}\rho\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega\\
\int\limits_{-\infty}^{\infty}\left(\frac{\omega^2}{c^2}\Phi\left(\mathbf{x},\omega\right)+\nabla^2\Phi\left(\mathbf{x},\omega\right)\right)e^{-i\omega t}d\omega &= -4\pi G_N\int\limits_{-\infty}^{\infty}\rho\left(\mathbf{x},\omega\right)e^{-i\omega t}d\omega
\end{aligned}$$

We see that the result is just the Fourier transform of the equation

$$\left(\nabla^2+k^2\right)\Phi\left(\mathbf{x},\omega\right)=-4\pi G_N\rho\left(\mathbf{x},\omega\right) \tag{1}$$

where we define $k\equiv\frac{\omega}{c}$. It is the completeness relation for Fourier transforms that guarantees we can invert the transform. See my notes on *Complex analysis* for a proof of the completeness relation.

The resulting equation is the *Helmholz equation*.

2.2 The homogeneous equation

With the sole boundary at infinity and no sources, the equation

$$\Box\phi=0$$

is not hard to solve. After the Fourier transform, we have the homogeneous Helmholz equation,

$$\left(\nabla^2+k^2\right)\phi\left(\mathbf{x},\omega\right)=0$$

Now, since there are no sources or boundaries, the system is spherically symmetric. As a result, there can be no angular dependence in ψ and the equation reduces to

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+k^2\phi &= 0\\
\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+k^2\phi &= 0\\
\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\phi\right)+k^2\phi &= 0\\
\frac{\partial^2}{\partial r^2}\left(r\phi\right)+k^2\left(r\phi\right) &= 0
\end{aligned}$$

and this last is just the one dimensional harmonic equation for the product, $r\phi$, with solution

$$r\phi=Ae^{ikr}+Be^{-ikr}$$

Finally, dividing by r,

$$\phi=\frac{Ae^{ikr}}{r}+\frac{Be^{-ikr}}{r}$$

which holds everywhere except $r=0$. In fact, we already know that the $\frac{1}{r}$ gives us the electrostatic Green's function. Since e^{ikr} approaches unity at $r=0$, and the same happens here. Specifically, we compute

$$\begin{aligned}
\left(\nabla^2+k^2\right)\frac{e^{\pm ikr}}{r} &= \boldsymbol{\nabla}\cdot\left(\frac{\pm ike^{\pm ikr}}{r}\hat{\mathbf{r}}+e^{\pm ikr}\boldsymbol{\nabla}\frac{1}{r}\right)+\frac{k^2e^{\pm ikr}}{r} \\
&= \boldsymbol{\nabla}\cdot\left(\frac{\pm ike^{\pm ikr}}{r}\hat{\mathbf{r}}\right)+\boldsymbol{\nabla}e^{\pm ikr}\cdot\boldsymbol{\nabla}\frac{1}{r}+e^{\pm ikr}\nabla^2\frac{1}{r}+\frac{k^2e^{\pm ikr}}{r} \\
&= -\frac{k^2e^{\pm ikr}}{r}-\frac{\pm ike^{\pm ikr}}{r^2}+\frac{\pm 2ike^{\pm ikr}}{r^2}-\frac{\pm ike^{\pm ikr}}{r^2} \\
&\quad +e^{\pm ikr}\nabla^2\frac{1}{r}+\frac{k^2e^{\pm ikr}}{r} \\
&= e^{\pm ikr}\nabla^2\frac{1}{r} \\
&= -4\pi e^{\pm ikr}\delta^3(\mathbf{x})
\end{aligned}$$

where we have used $\boldsymbol{\nabla}\cdot()=\hat{\mathbf{r}}\cdot\frac{d}{dr}()$ everywhere except for

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\hat{\mathbf{r}} &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\cdots\right)\frac{\hat{\mathbf{i}}x+\cdots}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{3}{\sqrt{x^2+y^2+z^2}}-\frac{1}{2}\frac{\mathbf{x}}{r^3}\cdot\left(2\hat{\mathbf{i}}x+\cdots\right) \\
&= \frac{3}{r}-\frac{\mathbf{x}}{r^3}\cdot\left(\hat{\mathbf{i}}x+\cdots\right) \\
&= \frac{3}{r}-\frac{r^2}{r^3} \\
&= \frac{2}{r}
\end{aligned}$$

Notice that $e^{\pm ikr}\delta^3(\mathbf{x})$ is the same as $\delta^3(\mathbf{x})$ since the Dirac delta forces $r=0$. Therefore,

$$\left(\nabla^2+k^2\right)\frac{e^{\pm ikr}}{r} = -4\pi\delta^3(\mathbf{x})$$

2.3 Green's function for the d'Alembertian

Now we need the Fourier transform of the equation for the Green function,

$$\Box G(\mathbf{x},t;\mathbf{x}',t')=-4\pi G_N\delta^3(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$

Because we have spherical symmetry, $G(\mathbf{x},t;\mathbf{x}',t')$ can depend only on

$$\begin{aligned}
R &= |\mathbf{x}-\mathbf{x}'| \\
\tau &= |t-t'|
\end{aligned}$$

and we use the Fourier transform of the τ-dependence,

$$G(R,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(R,\omega)\,e^{-i\omega\tau}d\omega$$

We also need the Fourier transform of the δ-function source. Using the completeness relation for the Fourier modes,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega$$

and substituting, the equation becomes

$$\begin{aligned}
\Box\left(\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}G\left(R,\omega\right)e^{-i\omega\tau}d\omega\right) &= -4\pi G_N\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\omega\tau}d\omega \\
\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left(\nabla^2+k^2\right)G\left(R,\omega\right)e^{-i\omega\tau}d\omega &= -4\pi G_N\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}e^{-i\omega\tau}d\omega \\
\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\left(\nabla^2+k^2\right)G\left(R,\omega\right)+\frac{4\pi G_N}{\sqrt{2\pi}}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right]e^{-i\omega\tau}d\omega &= 0
\end{aligned}$$

Inverting the transform, we need

$$\left(\nabla^2+k^2\right)G\left(R,\omega\right)=-4\pi G_N\left[\frac{1}{\sqrt{2\pi}}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\right]$$

and this is exactly the result we have for the homogeneous solution if we set

$$G_{\pm}\left(R,\omega\right)=\frac{G_N}{\sqrt{2\pi}}\frac{e^{\pm ikr}}{r}$$

We now just need to take the Fourier transform,

$$\begin{aligned}
G_{\pm}\left(R,\tau\right) &= \frac{G_N}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}G\left(R,\omega\right)e^{-i\omega\tau}d\omega \\
&= \frac{G_N}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{e^{\pm ikR}}{R}e^{-i\omega\tau}d\omega \\
&= \frac{G_N}{2\pi R}\int\limits_{-\infty}^{\infty}e^{-i\omega\tau\pm ikR}d\omega \\
&= \frac{G_N}{2\pi R}\int\limits_{-\infty}^{\infty}e^{-i\omega\left(\tau\mp\frac{R}{c}\right)}d\omega \\
&= \frac{G_N}{R}\delta\left(\tau\mp\frac{R}{c}\right)
\end{aligned}$$

where we have used $k=\frac{\omega}{c}$. Replacing R and τ, we have the *retarded and advanced Green functions*,

$$G_{\pm}\left(\mathbf{x},t;\mathbf{x}',t'\right) = \frac{G_N}{|\mathbf{x}-\mathbf{x}'|}\delta\left(|t-t'|\mp\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

The retarded Green function is

$$\begin{aligned}
G_{+}\left(\mathbf{x},t;\mathbf{x}',t'\right) &= \frac{G_N}{|\mathbf{x}-\mathbf{x}'|}\delta\left(|t-t'|-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \\
&= \frac{G_N}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}-t'\right) \\
&= \frac{G_N}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right)
\end{aligned}$$

This includes a time delay by the amount of time it takes a wave to propagate from $\mathbf{x}'$ to $\mathbf{x}$.

3 Solutions with sources

Finally, suppose we have a localized source, $\rho(\mathbf{x}', t')$ and we wish to solve the full equation,

$$\Box\Phi = -4\pi G_N \rho(\mathbf{x}, t)$$

This has the immediate solution

$$\begin{aligned}
\Phi &= \int G(\mathbf{x}, t; \mathbf{x}', t')\, \rho(\mathbf{x}', t')\, d^3x' dt' \\
&= -4\pi G_N \int \frac{1}{|\mathbf{x}-\mathbf{x}'|} \delta\left(t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right) \rho(\mathbf{x}', t')\, d^3x' dt' \\
&= -4\pi G_N \int \frac{\rho\left(\mathbf{x}', \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right)}{|\mathbf{x}-\mathbf{x}'|} d^3x' dt'
\end{aligned}$$

which is just like our static solution except that the effect of $\rho(\mathbf{x}')$ on the field at $\mathbf{x}$ and time t, is not $\rho(\mathbf{x}', t)$ but instead it is what ρ was at the earlier time $t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$. The effect is delayed by the time $\frac{|\mathbf{x}-\mathbf{x}'|}{c}$ that it takes light to propagate from $\mathbf{x}'$ to $\mathbf{x}$. For simplicity, for any source $f(\mathbf{x}', t')$, we define the retarded time notation,

$$[f(\mathbf{x}', t')]_{ret} \equiv f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

4 Electromagnetism

4.1 Energy and momentum

These originate as abstractions from electromagnetic forces as given by the Lorentz force law, but develop measurable properties of their own. The most notable is the propagation of electromagnetic energy and momentum as radiation. The fields satisfy the Maxwell equations,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho(\mathbf{x}, t)}{\epsilon_0} \\
\boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\
\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 \\
\boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} &= \mu_0 \mathbf{J}(\mathbf{x}, t)
\end{aligned}$$

Rather than dynamical variables parameterized only by time such as $\mathbf{x}(t)$ and $\mathbf{p}(t)$, the Maxwall fields depend on position as well

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}(\mathbf{x}, t) \\
\mathbf{B} &= \mathbf{B}(\mathbf{x}, t)
\end{aligned}$$

We start by comparing these to particle properties, then develop some generalizations.

To understand the relationship of a particle described by $\mathbf{x}(t)$ and $\mathbf{p}(t)$ to fields described by $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ it is important to realize that $\mathbf{x}(t)$ is *not* the same as the coordinate $\mathbf{x} = (x, y, z)$ or $\mathbf{x} = (r, \theta, \varphi)$. This becomes clear when we write the action functionals for the two cases. For a particle, we have

$$S = \int_C \left[\frac{1}{2} m \dot{\mathbf{x}}^2(t) - V(\mathbf{x}(t))\right] dt$$

Here the integral is over the *curve* $C = \{\mathbf{x}(t)\}$, which is the entire collection of points occupied by the physical system.

For the electromagnetic field, we work in terms of the potentials. Rewriting the electric and magnetic fields in terms of the scalar and vector potentials, $\Phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$,

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \end{aligned}$$

the homogeneous Maxwell equations are automatically satisfied, while the remaing two equations become wave equations

$$\begin{aligned} -\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \boldsymbol{\nabla}^2\Phi &= -\frac{\rho}{\epsilon_0} \\ -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \boldsymbol{\nabla}^2\mathbf{A} &= -\mu_0 \mathbf{J} \end{aligned}$$

provided we choose the potentials to satisfy the Lorentz gauge condition, $\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0$. This choice does not exhaust the gauge freedom since we may add the gradient of any function with vanishing Laplacian. We may choose this function to eliminate Φ, so the only degrees of freedom are those of $\mathbf{A}$ with $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

When we consider the effect of electromagnetic fields on the energy and momentum of charges, we find an extension of the conservation of energy by including an energy density

$$w = \frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

for the fields, and momentum density

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$$

Energy-momentum density 4-vector:

$$(w, \mathbf{S}) = \frac{1}{4\pi}\left(\frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), c\left(\mathbf{E} \times \mathbf{B}\right)\right)$$

4.2 Relativistic action for a particle

The relavitistic action for a particle in a potential V may be written as

$$S = \frac{1}{c}\int e^{V/mc^2}\left(-u^\alpha u_\alpha\right)^{1/2} d\tau$$

(see Dynamics in special relativity). For a free particle, this is proportional to the magnitude of the 4-velocity. For electromagnetism, we have a current density instead,

$$J^\alpha = (w, \mathbf{S}) = \frac{1}{4\pi}\left(\frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), c\left(\mathbf{E} \times \mathbf{B}\right)\right)$$

but we may still take its magnitude. The norm of the current density is

$$\begin{aligned} J^\alpha J_\alpha &= \frac{1}{16\pi^2}\left(-\frac{c^2}{4}\left(\mathbf{E}^2 + \mathbf{B}^2\right)^2 + c^2\left(\mathbf{E} \times \mathbf{B}\right) \cdot \left(\mathbf{E} \times \mathbf{B}\right)\right) \\ &= -\frac{1}{4}\frac{c^2}{16\pi^2}\left(\left(\mathbf{E}^2\mathbf{E}^2 + 2\mathbf{E}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{B}^2\right) - 4\mathbf{B} \cdot \left(\left(\mathbf{E} \times \mathbf{B}\right) \times \mathbf{E}\right)\right) \\ &= \frac{c^2}{16\pi^2}\left(\mathbf{E}^2\mathbf{E}^2 + 2\mathbf{E}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{B}^2 - 4\mathbf{B} \cdot \left(\mathbf{B}\left(\mathbf{E}^2\right) - \mathbf{E}\left(\mathbf{E} \cdot \mathbf{B}\right)\right)\right) \\ &= \frac{c^2}{16\pi^2}\left(\mathbf{E}^2\mathbf{E}^2 + 2\mathbf{E}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{B}^2 - 4\mathbf{B}^2\mathbf{E}^2 - \left(\mathbf{E} \cdot \mathbf{B}\right)^2\right) \end{aligned}$$

For freely propagaging electromagnetic fields, $\mathbf{E}\cdot\mathbf{B}=0$, so

$$\begin{aligned}
\frac{1}{c}\sqrt{J^\alpha J_\alpha} &= \frac{1}{4\pi}\sqrt{\left(\mathbf{E}^2\mathbf{E}^2-2\mathbf{E}^2\mathbf{B}^2+\mathbf{B}^2\mathbf{B}^2\right)} \\
&= \frac{1}{4\pi}\left(\mathbf{E}^2-\mathbf{B}^2\right)
\end{aligned}$$

This has units of energy density, so we may take Lagrangian to be

$$L=\frac{1}{2}\int\left(\mathbf{E}^2-\mathbf{B}^2\right)d^3x$$

Finally, integrating over time, we find the action

$$S = \frac{1}{2}\int\left(\mathbf{E}^2-\mathbf{B}^2\right)d^4x$$

Notice that this expression depends only on the fields and therefore is independent of the gauge choice.

4.2.1 Variation in the Lorentz gauge

To see that this resembles $T-V$, expand in terms of the potentials. Remembering that our gauge choice gives $\frac{\partial\Phi}{\partial t}=0$,

$$S = \frac{1}{2}\int\left(\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial\mathbf{A}}{\partial t}-(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{A})\right)d^4x$$

We may think of the time derivatives, $\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial\mathbf{A}}{\partial t}$, as giving the kinetic part and the spatial terms, $(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{A})$ as the potential. To include the source current, $\mathbf{J}$, we look at the potential energy density. The energy of a charge q in a scalar potential Φ is just $q\Phi$. As a density, this generalizes to

$$\rho\Phi+\mathbf{J}\cdot\mathbf{A}$$

which reduces to $\mathbf{J}\cdot\mathbf{A}$ when $\Phi=0$. Subtracting this contribution to the potential, we have

$$S = \int\left(\frac{1}{2}\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial\mathbf{A}}{\partial t}-\frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{A})-\mathbf{J}\cdot\mathbf{A}\right)d^4x$$

Now vary the potential.

$$\begin{aligned}
0 &= \delta S \\
&= \int\left(\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial\delta\mathbf{A}}{\partial t}-(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\delta\mathbf{A})-\mathbf{J}\cdot\delta\mathbf{A}\right)d^4x \\
&= \int\left(\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial\delta\mathbf{A}}{\partial t}-(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\delta\mathbf{A})-\mathbf{J}\cdot\delta\mathbf{A}\right)d^4x
\end{aligned}$$

We may write the first term as

$$\frac{\partial\mathbf{A}}{\partial t}\cdot\frac{\partial\delta\mathbf{A}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{A}}{\partial t}\cdot\delta\mathbf{A}\right)-\frac{\partial^2\mathbf{A}}{\partial t^2}\cdot\delta\mathbf{A}$$

The second term takes a little more work,

$$\begin{aligned}
-(\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\delta\mathbf{A}) &= -\varepsilon^{ijk}\varepsilon^{imn}\partial_j A_k\partial_m\delta A_n \\
&= \partial_m\left(-\varepsilon^{ijk}\varepsilon^{imn}\partial_j A_k\delta A_n\right)-\left(-\varepsilon^{ijk}\varepsilon^{imn}\left(\partial_m\partial_j A_k\right)\delta A_n\right) \\
&= \partial_m\left(\varepsilon^{min}\left(\boldsymbol{\nabla}\times\mathbf{A}\right)_i\delta A_n\right)+\left(\delta^{jm}\delta^{kn}-\delta^{jn}\delta^{km}\right)\left(\partial_m\partial_j A_k\right)\delta A_n \\
&= \boldsymbol{\nabla}\cdot\left((\boldsymbol{\nabla}\times\mathbf{A})\times\delta\mathbf{A}\right)+\left(\left(\nabla^2\mathbf{A}\right)\cdot\delta\mathbf{A}-\delta\mathbf{A}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)\right)
\end{aligned}$$

The total time derivative may be integrated to the timelike boundary, while the divergence may be integrated to the spatial boundary. The gauge condition lets us drop the $\boldsymbol{\nabla}\cdot\mathbf{A}$ term. This leaves us with

$$\begin{aligned} 0 &= \int\left(-\frac{\partial^2\mathbf{A}}{\partial t^2}\cdot\delta\mathbf{A}+\left(\nabla^2\mathbf{A}\right)\cdot\delta\mathbf{A}-\mathbf{J}\cdot\delta\mathbf{A}\right)d^4x \\ &= \int\left(-\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla^2\mathbf{A}-\mathbf{J}\right)\cdot\delta\mathbf{A}d^4x \end{aligned}$$

and therefore we recover the field equation,

$$-\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla^2\mathbf{A}=\mathbf{J}$$

This is a common form of bosonic wave equation.

4.2.2 Gauge independent variation

We can do the same calculation without choosing a gauge. Adding sources to the action we have

$$S=\int\left(\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\rho\Phi-\mathbf{J}\cdot\mathbf{A}\right)d^4x$$

where

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\Phi-\frac{\partial\mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} \end{aligned}$$

Expanding the action in terms of the potentials, we have

$$\begin{aligned} S &= \int\left(\frac{1}{2}\left(\mathbf{E}^2-\mathbf{B}^2\right)-\rho\Phi-\mathbf{J}\cdot\mathbf{A}\right)d^4x \\ &= \int\left(\frac{1}{2}\left(-\boldsymbol{\nabla}\Phi-\frac{\partial\mathbf{A}}{\partial t}\right)\cdot\left(-\boldsymbol{\nabla}\Phi-\frac{\partial\mathbf{A}}{\partial t}\right)-\frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\cdot\left(\boldsymbol{\nabla}\times\mathbf{A}\right)-\rho\Phi-\mathbf{J}\cdot\mathbf{A}\right)d^4x \end{aligned}$$

Variation of Φ and $\mathbf{A}$ gives

$$\begin{aligned} 0 &= \delta S \\ &= \int\left(\frac{1}{2}2\left(-\boldsymbol{\nabla}\Phi-\frac{\partial\mathbf{A}}{\partial t}\right)\cdot\left(-\boldsymbol{\nabla}\delta\Phi-\frac{\partial\delta\mathbf{A}}{\partial t}\right)-\frac{1}{2}2\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\cdot\left(\boldsymbol{\nabla}\times\delta\mathbf{A}\right)-\rho\delta\Phi-\mathbf{J}\cdot\delta\mathbf{A}\right)d^4x \\ &= \int\left(\boldsymbol{\nabla}\cdot\left[\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\delta\Phi\right]-\left[\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\right]\delta\Phi\right)d^4x \\ &\quad+\int\left(\frac{\partial}{\partial t}\left[\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\cdot\delta\mathbf{A}\right]-\left[\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\right]\cdot\delta\mathbf{A}\right)d^4x \\ &\quad+\int\left(-\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\cdot\left(\boldsymbol{\nabla}\times\delta\mathbf{A}\right)-\rho\delta\Phi-\mathbf{J}\cdot\delta\mathbf{A}\right)d^4x \\ &= \int\left[\boldsymbol{\nabla}\cdot\left[\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\delta\Phi\right]+\frac{\partial}{\partial t}\left[\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\cdot\delta\mathbf{A}\right]\right]d^4x \\ &\quad+\int\left(-\left[\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\right]\delta\Phi-\left[\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\Phi+\frac{\partial\mathbf{A}}{\partial t}\right)\right]\cdot\delta\mathbf{A}\right)d^4x \\ &\quad+\int\left(\boldsymbol{\nabla}\cdot\left(\left(\boldsymbol{\nabla}\times\mathbf{A}\right)\times\delta\mathbf{A}\right)+\left(\nabla^2\mathbf{A}\right)\cdot\delta\mathbf{A}-\delta\mathbf{A}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}\right)-\rho\delta\Phi-\mathbf{J}\cdot\delta\mathbf{A}\right)d^4x \end{aligned}$$

$$\begin{aligned} &= \int \left[\boldsymbol{\nabla} \cdot \left[\left(\boldsymbol{\nabla}\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) \delta\Phi + (\boldsymbol{\nabla} \times \mathbf{A}) \times \delta\mathbf{A} \right] + \frac{\partial}{\partial t} \left[\left(\boldsymbol{\nabla}\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) \cdot \delta\mathbf{A} \right] \right] d^4x \\ &\quad + \int \left(-\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla}\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) - \rho \right) \delta\Phi d^4x \\ &\quad + \int \left(-\frac{\partial}{\partial t} \left(\boldsymbol{\nabla}\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) + \nabla^2\mathbf{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \mathbf{J} \right) \cdot \delta\mathbf{A} d^4x \end{aligned}$$

The first terms may be integrated to the boundary, where the variations vanish. The arbitrariness of $\delta\Phi$ and $\delta\mathbf{A}$ then gives

$$\begin{aligned} -\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla}\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) - \rho &= 0 \\ -\frac{\partial}{\partial t} \left(\boldsymbol{\nabla}\Phi + \frac{\partial \mathbf{A}}{\partial t} \right) + \nabla^2\mathbf{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \mathbf{J} &= 0 \end{aligned}$$

Recalling that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\nabla^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A})$, and replacing the potentials with the fields, the first becomes

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho \\ \frac{\partial \mathbf{E}}{\partial t} - \boldsymbol{\nabla} \times \mathbf{B} &= \mathbf{J} \end{aligned}$$

5 Matter waves

As we have seen, various forms of the relativistic wave equation,

$$-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \nabla^2\Phi = -J$$

arise frequently in classical field theory. The wave operator,

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

called the d'Alembertian is the natural generalization of the Laplacian to the 4-spacetime dimensions.

It was natural for the ideas of wave-particle duality in the early 20^{th} century to make use of the d'Alembertian. These efforts lead naturally to the Klein-Gordon equation.

5.1 The Klein-Gordon equation

The deBroglie wavelength and the Planck relation, together with the relativistic relationship between energy and momentum, relate the particle properties of energy and momentum to the wave properties of frequency and wavelength,

$$\begin{aligned} E &= \hbar\omega \\ \mathbf{p} &= \hbar\mathbf{k} \end{aligned}$$

The 4-momentum of a particle is given by

$$\begin{aligned} p^\alpha &= mu^\alpha \\ &= m\gamma(c, \mathbf{v}) \\ &= \left(\frac{E}{c}, \mathbf{p} \right) \end{aligned}$$

and the invariant norm of the four momentum is

$$\begin{aligned}\eta_{\alpha\beta}p^\alpha p^\beta &= p^\alpha p_\alpha \\ &= -\left(p^0\right)^2 + \mathbf{p}^2 \\ &= -\frac{E^2}{c^2} + \mathbf{p}^2\end{aligned}$$

By construction, this quantity is invariant under Lorentz transformations. We see this explicityly by using the norm of the 4-velocity, $u^\alpha u_\alpha = -c^2$ to writh

$$\begin{aligned}\eta_{\alpha\beta}p^\alpha p^\beta &= m^2 u^\alpha u_\alpha \\ &= -m^2 c^2\end{aligned}$$

which is clearly independent of inertial frame of reference. Equating the two expressions gives the mass-energy relation,

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4$$

Now suppose the electron is described by a plane wave,

$$\psi = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

Then we may recover the wave number and frequency by differentiation,

$$\begin{aligned}\boldsymbol{\nabla}^2\psi &= \boldsymbol{\nabla}^2\left[Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\right] \\ &= -\mathbf{k}^2\psi \\ \frac{\partial^2}{\partial t^2}\psi &= \frac{\partial^2}{\partial t^2}\left[Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\right] \\ &= -\omega^2\psi\end{aligned}$$

Multiplying each derivative by $i\hbar$, we have the energy and momentum,

$$\begin{aligned}\left(i\hbar\boldsymbol{\nabla}\right)^2\psi &= \hbar^2\mathbf{k}^2\psi \\ &= \mathbf{p}^2\psi \\ \left(i\hbar\frac{\partial}{\partial t}\right)^2\psi &= \hbar^2\omega^2\psi \\ &= E^2\psi\end{aligned}$$

Substituting these operators,

$$\begin{aligned}p_\alpha = \left(-\frac{E}{c}, \mathbf{p}\right) &= -i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right) \\ &= -i\hbar\frac{\partial}{\partial x^\alpha}\end{aligned}$$

into the energy-momentum relation,

$$\begin{aligned}E^2 &= \mathbf{p}^2 c^2 + m^2 c^4 \\ \left(i\hbar\frac{\partial}{\partial t}\right)^2 &= \left(i\hbar\boldsymbol{\nabla}\right)^2 c^2 + m^2 c^4\end{aligned}$$

and allowing this operator relationship to act on a "wave function", ψ,

$$\begin{aligned}
-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} &= -\hbar^2 c^2 \boldsymbol{\nabla}^2 \psi + m^2 c^4 \psi \\
-\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} + \boldsymbol{\nabla}^2 \psi &= \frac{m^2 c^2}{\hbar^2} \psi
\end{aligned}$$

Writing this in terms of the d'Alembertian,

$$\Box \psi = \frac{m^2 c^2}{\hbar^2} \psi$$

we have the *Klein-Gordon equation*. Because of the Planck and deBroglie relationships, it also describes particle-like energy and momentum. Indeed, the plane-wave solutions may be written as

$$\psi = A e^{\frac{i}{\hbar}(\mathbf{p} \cdot \mathbf{x} - Et)}$$

The Klein-Gordon equation first appears in Schrödinger's notes in 1925 before being published the next year first by Oskar Klein and Walter Gordon, but also the same year by Vladimir Fock, Johann Kudar, Théophile de Donder and Frans-H. van den Dungen, and Louis de Broglie. It is the obvious relativistic generalization of the Schrödinger equation but fails to describe electron spin. Additionally, because the equation is second order in time derivatives, it requires both initial position and velocity specifications, and this is forbidden by the uncertainty principle. Finally, the equation leads to negative probability states.

5.2 The Schrödinger equation

In 1925, Schrödinger took a different approach. The problems arising from the second order time derivatives may be avoided by first solving for the energy, then taking a non-relativistic approximation. We may then also add a potential to the energy

$$\begin{aligned}
E &= \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} + V \\
&= mc^2 \sqrt{1 + \frac{\mathbf{p}^2}{m^2 c^2}} + V
\end{aligned}$$

For $v \ll c$ we may expand $\sqrt{1 + \frac{\mathbf{p}^2}{m^2 c^2}}$ in a Taylor series,

$$\begin{aligned}
\sqrt{1 + \frac{\mathbf{p}^2}{m^2 c^2}} &= 1 + \frac{\mathbf{p}^2}{2m^2 c^2} + \cdots \\
&\approx 1 + \frac{\mathbf{p}^2}{2m^2 c^2}
\end{aligned}$$

so the non-relativistic version is

$$E = mc^2 \left(1 + \frac{\mathbf{p}^2}{2m^2 c^2}\right) + V$$

Making the same operator substitutions that led us to the Klein-Gordon equation, and allowing it to operate on a function, ϕ, gives

$$i\hbar \frac{\partial \phi}{\partial t} = mc^2 \phi - \frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \phi + V\phi$$

The constant mass term may be removed by the replacement

$$\phi = \psi e^{-\frac{i}{\hbar} mc^2 t}$$

Then we find

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left(\psi e^{-\frac{i}{\hbar}mc^2t}\right) &= mc^2\psi e^{-\frac{i}{\hbar}mc^2t} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\left(\psi e^{-\frac{i}{\hbar}mc^2t}\right) + V\psi e^{-\frac{i}{\hbar}mc^2t} \\
i\hbar\left(\frac{\partial\psi}{\partial t}e^{-\frac{i}{\hbar}mc^2t} - \frac{i}{\hbar}mc^2\psi e^{-\frac{i}{\hbar}mc^2t}\right) &= \left(mc^2\psi - \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + V\psi\right)e^{-\frac{i}{\hbar}mc^2t}
\end{aligned}
$$

resulting in the familiar form of the Schrödinger equation,

$$
i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + V\psi
$$