1 Bhabha scattering (1936)

Consider the scattering of an electron and a positron. We have already computed the annihilation of an electron/positron in finding the cross-section for producing muon pairs. It is also possible for the final particles to be an electron-positron pair. In addition, the initial particles may exchage a photon without annihilating. Since the particles are distinguishable, we have only the t-channel diagram for this, without the crossed diagram. Let the in and out electrons have momenta p, p', respectively and the initial and final positrons have momenta k, k'. Let the angle between the initial and final electrons be θ.

The general calculation is lengthy; here we restrict our attention to the ultrarelativistic case, where we can treat the electron and positron as massless. This means we neglect terms of order $\frac{m^2}{E^2}$. For GeV or TeV colliders, this term is of order 10^{-8} or 10^{-14}, so we incur no great error. We may drop m whenever it occurs additively.

1.1 The scattering matrix

We have two Feynman diagrams, so the matrix element is the sum

$$i\mathscr{M}_{fi} = i\mathscr{M}^s_{fi} + i\mathscr{M}^t_{fi}$$

where s and t refer to the photon channel. The first, s-channel, diagram describes the annihilation-creation. The matrix element is the same as we use for the muon problem, except the particle masses are all equal, and here we want to call p, k the incoming momenta and p', k' the outgoing momenta. Therefore, instead of

$$i\mathscr{M}^s_{fi} = \bar{v}\left(p'\right)\left(-ie\gamma^\alpha\right)u\left(p\right)\frac{-i}{q^2}\left(\eta_{\alpha\beta} - \left(1-\xi\right)\frac{q_\alpha q_\beta}{q^2}\right)\bar{u}\left(k\right)\left(-ie\gamma^\beta\right)v\left(k'\right)$$

we write

$$i\mathscr{M}^s_{fi} = \bar{v}\left(k\right)\left(-ie\gamma^\alpha\right)u\left(p\right)\frac{-i}{q^2}\left(\eta_{\alpha\beta} - \left(1-\xi\right)\frac{q_\alpha q_\beta}{q^2}\right)\bar{u}\left(p'\right)\left(-ie\gamma^\beta\right)v\left(k'\right)$$

that is, we just need to interchange $p' \leftrightarrow k$ in our previous result, and we have $q = p + k = p' + k'$. As before, the gauge terms vanish immediately. We replace $q^2 = s$, leaving

$$i\mathscr{M}^s_{fi} = \frac{ie^2}{s}\eta_{\alpha\beta}\bar{v}\left(k\right)\gamma^\alpha u\left(p\right)\bar{u}\left(p'\right)\gamma^\beta v\left(k'\right)$$

The second Feynman diagram has the matrix element

$$i\mathscr{M}^t_{fi} = \bar{u}\left(p'\right)\left(-ie\gamma^\alpha\right)u\left(p\right)\frac{-i}{q^2}\left(\eta_{\alpha\beta} - \left(1-\xi\right)\frac{q_\alpha q_\beta}{q^2}\right)\bar{v}\left(k\right)\left(-ie\gamma^\beta\right)v\left(k'\right)$$

where p, p' are the incoming and outgoing electron, respectively; k, k' are the incoming and outgoing proton (or any other fermion of charge Q, except the electron which would also require the u channel), respectively; $q = p - p' = k' - k$ is the t-channel 4-momentum; ξ allows a choice of gauge.

The gauge term does not immediately drop in this case. We have:

$$\begin{aligned}
\bar{u}\left(p'\right)\left(-ie\gamma^\alpha\right)u\left(p\right)\frac{-i}{q^2}\left(-\left(1-\xi\right)\frac{q_\alpha q_\beta}{q^2}\right)\bar{v}\left(k\right)\left(-ie\gamma^\beta\right)v\left(k'\right) &= \frac{i}{t^2}\left(1-\xi\right)e^2\bar{u}\left(p'\right)\left(i\not{q}\right)u\left(p\right)\bar{v}\left(k\right)\left(i\not{q}\right)v\left(k'\right) \\
&= \frac{i}{t^2}\left(1-\xi\right)e^2\bar{u}\left(p'\right)\left(i\not{p} - i\not{p}'\right)u\left(p\right)\bar{v}\left(k\right)\left(i\not{k}' - i\not{k}\right)v\left(k'\right) \\
&= \frac{i}{t^2}\left(1-\xi\right)e^2\bar{u}\left(p'\right)2mu\left(p\right)\bar{v}\left(k\right)2mv\left(k'\right) \\
&= \frac{4im^2e^2}{t^2}\left(1-\xi\right)e^2\bar{u}\left(p'\right)u\left(p\right)\bar{v}\left(k\right)v\left(k'\right)
\end{aligned}$$

where we use the Dirac and conjugate Dirac equations in the middle step. The matrix element becomes

$$i\mathscr{M}^t_{fi} = \frac{ie^2}{t}\eta_{\alpha\beta}\bar{u}\left(p'\right)\gamma^\alpha u\left(p\right)\bar{u}\left(k'\right)\gamma^\beta u\left(k\right) - \frac{4im^2e^2}{t^2}\left(1-\xi\right)e^2\bar{u}\left(p'\right)u\left(p\right)\bar{v}\left(k\right)v\left(k'\right)$$

where we set $q^2 = t$. We should get the same result in any gauge, so we could just set $\xi = 1$, but in any case the term of of order m^2 and we drop it.

The resulting matrix element is

$$\begin{aligned} i\mathscr{M}_{fi} &= i\mathscr{M}^s_{fi} + i\mathscr{M}^t_{fi} \\ &= \frac{ie^2}{s}\eta_{\alpha\beta}\bar{v}(k)\gamma^\alpha u(p)\bar{u}(p')\gamma^\beta v(k') + \frac{ie^2}{t}\eta_{\mu\nu}\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma^\nu v(k') \end{aligned}$$

Squaring the matrix element leads to four terms,

$$\left|i\mathscr{M}_{fi}\right|^2 = \left|i\mathscr{M}^s_{fi}\right|^2 + \mathscr{M}^s_{fi}\mathscr{M}^{t*}_{fi} + \mathscr{M}^t_{fi}\mathscr{M}^{s*}_{fi} + \left|i\mathscr{M}^t_{fi}\right|^2$$

We work them out one at a time.

1.2 Matrix squared: s channel

We already have $\left|i\mathscr{M}^s_{fi}\right|^2$ from the muon case. Averaging over initial spins and summing over final spins, we found

$$\begin{aligned} \frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}^s_{fi}\right|^2 &= \frac{8e^4}{s^2}\left(\left(\left(p\cdot p'\right)\left(k\cdot k'\right) - \left(k'\cdot p'\right)(k\cdot p) + \left(p'\cdot k\right)\left(p\cdot k'\right) - m^2(k\cdot p)\right) + (k\cdot p)\left(k'\cdot p'\right) + m^2\left(k'\cdot p'\right) + 2m^2(k\cdot p) + 2\right. \\ &= \frac{8e^4}{s^2}\left(\left(p\cdot p'\right)\left(k\cdot k'\right) + \left(p'\cdot k\right)\left(p\cdot k'\right) + m^2\left(k'\cdot p'\right) + m^2(k\cdot p) + 2m^4\right) \end{aligned}$$

where some simplification occurs because of the equal masses.

1.3 Matrix squared: t channel

For the final term, we have

$$\begin{aligned} \frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}^t_{fi}\right|^2 &= \frac{1}{4}\sum_{all\,spins}\left(\frac{ie^2}{t}\eta_{\mu\nu}\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma^\nu v(k')\right)\left(-\frac{ie^2}{t}\eta_{\rho\sigma}\bar{v}(k')\gamma^\rho v(k)\bar{u}(p)\gamma^\sigma u(p')\right) \\ &= \sum_{all\,spins}\frac{e^4}{4t^2}\eta_{\mu\nu}\eta_{\rho\sigma}\bar{u}(p')\gamma^\mu u(p)\bar{v}(k)\gamma^\nu v(k')\bar{v}(k')\gamma^\rho v(k)\bar{u}(p)\gamma^\sigma u(p') \\ &= \sum_{all\,spins}\frac{e^4}{4t^2}\eta_{\mu\nu}\eta_{\rho\sigma}\bar{u}_a(p')\gamma^\mu_{ab}u_b(p)\bar{v}_c(k)\gamma^\nu_{cd}v_d(k')\bar{v}_e(k')\gamma^\rho_{ef}v_f(k)\bar{u}_g(p)\gamma^\sigma_{gh}u_h(p') \\ &= (-1)^{14}\sum_{all\,spins}\frac{e^4}{4t^2}\eta_{\mu\nu}\eta_{\rho\sigma}\gamma^\mu_{ab}u_b(p)\bar{u}_g(p)\gamma^\sigma_{gh}u_h(p')\bar{u}_a(p')\gamma^\nu_{cd}v_d(k')\bar{v}_e(k')\gamma^\rho_{ef}v_f(k)\bar{v}_c(k) \\ &= \frac{e^4}{4t^2}\eta_{\mu\nu}\eta_{\rho\sigma}\gamma^\mu_{ab}(\not{p}+m)_{bg}\gamma^\sigma_{gh}\left(\not{p}'+m\right)_{ha}\gamma^\nu_{cd}\left(\not{k}'-m\right)_{de}\gamma^\rho_{ef}(\not{k}-m)_{fc} \\ &= \frac{e^4}{4t^2}\eta_{\mu\nu}\eta_{\rho\sigma}tr\left(\gamma^\mu(\not{p}+m)\gamma^\sigma\left(\not{p}'+m\right)\right)tr\left(\gamma^\nu\left(\not{k}'-m\right)\gamma^\rho(\not{k}-m)\right) \end{aligned}$$

where the factor $(-1)^{14}$ keeps track of the number of times we commute fermion fields. The traces give

$$\begin{aligned} tr\left(\gamma^\mu(\not{p}+m)\gamma^\sigma\left(\not{p}'+m\right)\right) &= tr\left(\gamma^\mu\not{p}\gamma^\sigma\not{p}' + m\gamma^\mu\gamma^\sigma\not{p}' + m\gamma^\mu\not{p}\gamma^\sigma + m^2\gamma^\mu\gamma^\sigma\right) \\ &= tr\left(\gamma^\mu\not{p}\gamma^\sigma\not{p}'\right) + m^2 tr\left(\gamma^\mu\gamma^\sigma\right) \\ &= p_\lambda p'_\tau tr\left(\gamma^\mu\gamma^\lambda\gamma^\sigma\gamma^\tau\right) + 4m^2\eta^{\mu\sigma} \\ &= 4p_\lambda p'_\tau\left(\eta^{\mu\lambda}\eta^{\sigma\tau} - \eta^{\mu\sigma}\eta^{\lambda\tau} + \eta^{\mu\tau}\eta^{\lambda\sigma}\right) + 4m^2\eta^{\mu\sigma} \\ &= 4p_\lambda p'_\tau\left(\eta^{\mu\lambda}\eta^{\sigma\tau} - \eta^{\mu\sigma}\eta^{\lambda\tau} + \eta^{\mu\tau}\eta^{\lambda\sigma}\right) + 4m^2\eta^{\mu\sigma} \\ &= 4\left(p^\mu p'^\sigma - \left(p\cdot p'\right)\eta^{\mu\sigma} + p'^\mu p^\sigma + m^2\eta^{\mu\sigma}\right) \\ tr\left(\gamma^\nu\left(\not{k}'-m\right)\gamma^\rho(\not{k}-m)\right) &= 4\left(k'^\nu k^\rho - \left(k\cdot k'\right)\eta^{\nu\rho} + k^\nu k'^\rho + m^2\eta^{\nu\rho}\right) \end{aligned}$$

Combining these results,

$$\begin{aligned}\frac{1}{4}\sum_{all\,spins}\left|\mathcal{M}^t_{fi}\right|^2 &= \frac{4e^4}{t^2}\eta_{\mu\nu}\eta_{\rho\sigma}\left(p^\mu p'^\sigma-\left(p\cdot p'\right)\eta^{\mu\sigma}+p'^\mu p^\sigma+m^2\eta^{\mu\sigma}\right)\left(k'^\nu k^\rho-\left(k\cdot k'\right)\eta^{\nu\rho}+k^\nu k'^\rho+m^2\eta^{\nu\rho}\right)\\ &= \frac{4e^4}{t^2}p_\nu p'_\rho\left(k'^\nu k^\rho-\left(k\cdot k'\right)\eta^{\nu\rho}+k^\nu k'^\rho+m^2\eta^{\nu\rho}\right)\\ &\quad -\frac{4e^4}{t^2}\left(p\cdot p'\right)\eta_{\nu\rho}\left(k'^\nu k^\rho-\left(k\cdot k'\right)\eta^{\nu\rho}+k^\nu k'^\rho+m^2\eta^{\nu\rho}\right)\\ &\quad +\frac{4e^4}{t^2}p'_\nu p_\rho\left(k'^\nu k^\rho-\left(k\cdot k'\right)\eta^{\nu\rho}+k^\nu k'^\rho+m^2\eta^{\nu\rho}\right)\\ &\quad +\frac{4m^2e^4}{t^2}\eta_{\nu\rho}\left(k'^\nu k^\rho-\left(k\cdot k'\right)\eta^{\nu\rho}+k^\nu k'^\rho+m^2\eta^{\nu\rho}\right)\end{aligned}$$

Performing the remaining contractions,

$$\begin{aligned}\frac{1}{4}\sum_{all\,spins}\left|\mathcal{M}^t_{fi}\right|^2 &= \frac{4e^4}{t^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)-\left(k\cdot k'\right)\left(p\cdot p'\right)+\left(p\cdot k\right)\left(p'\cdot k'\right)+m^2\left(p\cdot p'\right)\right)\\ &\quad -\frac{4e^4}{t^2}\left(p\cdot p'\right)\left(\left(k\cdot k'\right)-4\left(k\cdot k'\right)+\left(k\cdot k'\right)+4m^2\right)\\ &\quad +\frac{4e^4}{t^2}\left(\left(p'\cdot k'\right)\left(p\cdot k\right)-\left(k\cdot k'\right)\left(p\cdot p'\right)+\left(p'\cdot k\right)\left(p\cdot k'\right)+m^2\left(p\cdot p'\right)\right)\\ &\quad +\frac{4m^2e^4}{t^2}\left(\left(k\cdot k'\right)-4\left(k\cdot k'\right)+\left(k\cdot k'\right)+4m^2\right)\\ &= \frac{8e^4}{t^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)-\left(k\cdot k'\right)\left(p\cdot p'\right)+\left(p\cdot k\right)\left(p'\cdot k'\right)+m^2\left(p\cdot p'\right)+\left(m^2-\left(p\cdot p'\right)\right)\left(-\left(k\cdot k'\right)+2m^2\right)\right)\\ &= \frac{8e^4}{t^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)+\left(p\cdot k\right)\left(p'\cdot k'\right)-m^2\left(p\cdot p'\right)-m^2\left(k\cdot k'\right)+2m^4\right)\end{aligned}$$

1.4 Matrix squared: first cross term

Now consider the first of the cross terms. We need to make sure we've assigned the same momenta to the four particles. But for the s channel we've set p' as the initial positron momentum instead of the final.

$$\begin{aligned}\frac{1}{4}\sum_{all\,spins}\mathcal{M}^s_{fi}\mathcal{M}^{t*}_{fi} &= \frac{1}{4}\sum_{all\,spins}\left(\frac{ie^2}{s}\eta_{\alpha\beta}\bar{v}(k)\gamma^\alpha u(p)\bar{u}\left(p'\right)\gamma^\beta v\left(k'\right)\right)\left(-\frac{ie^2}{t}\eta_{\mu\nu}\bar{v}\left(k'\right)\gamma^\nu v(k)\bar{u}(p)\gamma^\mu u\left(p'\right)\right)\\ &= \frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\sum_{all\,spins}\bar{v}_a(k)\gamma^\alpha_{ab}u_b(p)\bar{u}_c\left(p'\right)\gamma^\beta_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\gamma^\nu_{ef}v_f(k)\bar{u}_g(p)\gamma^\mu_{gh}u_h\left(p'\right)\\ &= (-1)^{15}\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\sum_{all\,spins}\gamma^\alpha_{ab}u_b(p)\bar{u}_g(p)\gamma^\beta_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\gamma^\nu_{ef}v_f(k)\bar{v}_a(k)\gamma^\mu_{gh}u_h\left(p'\right)\bar{u}_c\left(p'\right)\\ &= -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\not{k}_{fa}\gamma^\alpha_{ab}\not{p}_{bg}\gamma^\mu_{gh}\not{p}'_{hc}\gamma^\beta_{cd}\not{k}'_{de}\gamma^\nu_{ef}\\ &= -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu\right)\end{aligned}$$

where we have dropped all mass terms. Notice that the fermion exchanges have introduced an overall sign.

Now evaluate the trace. First, we eliminate the free γ-matrices, using $tr\left(\eta_{\alpha\beta}\gamma^\alpha\gamma^\beta\right)=\frac{1}{2}tr\left(\eta_{\alpha\beta}\left\{\gamma^\alpha,\gamma^\beta\right\}\right)=4$. To do this, we must bring the two relevant gamma matrices next to each other. Also, notice that

$$\begin{aligned}\gamma^\alpha\not{p} &= p_\beta\gamma^\alpha\gamma^\beta\left(2p^\alpha-\not{p}\gamma^\alpha\right)\\ &= p_\beta\left(2\eta^{\alpha\beta}-\gamma^\beta\gamma^\alpha\right)\\ &= 2p^\alpha-\not{p}\gamma^\alpha\end{aligned}$$

and for any of p,k,p',k',

$$\begin{aligned}
\not{p}\not{p} &= p_\alpha p_\beta \gamma^\alpha \gamma^\beta \\
&= \frac{1}{2} p_\alpha p_\beta \left\{ \gamma^\alpha, \gamma^\beta \right\} \\
&= \frac{1}{2} p_\alpha p_\beta 2\eta^{\alpha\beta} \\
&= p^2 \\
&= 0
\end{aligned}$$

The trace is

$$\frac{1}{4}\sum_{all\,spins} \mathscr{M}^s_{fi}\mathscr{M}^{t*}_{fi} = -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{k}'\gamma^\nu\right)$$

Using conservation of momentum, we replace $k' = p+k-p'$,

$$\begin{aligned}
\frac{1}{4}\sum_{all\,spins} \mathscr{M}^s_{fi}\mathscr{M}^{t*}_{fi} &= -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \left(\not{p}+\not{k}-\not{p}'\right)\gamma^\nu\right) \\
&= -\frac{e^4}{4st}\left(T_p + T_k - T_{p'}\right)
\end{aligned}$$

where

$$\begin{aligned}
T_p &= \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{p}\gamma^\nu\right) \\
T_k &= \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{k}\gamma^\nu\right) \\
T_{p'} &= \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{p}'\gamma^\nu\right) \\
&= \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{p}'\gamma^\nu \not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta\right)
\end{aligned}$$

where we used the cyclic property on $T_{p'}$ to check that it is related to T_k by the cyclic substitution, $k \to p' \to p \to k$, and renaming the dummy Lorentz indices. We therefore only need to compute the first two.

For T_p,

$$\begin{aligned}
T_p &= \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \not{p}\gamma^\mu \not{p}'\gamma^\beta \not{p}\gamma^\nu\right) \\
&= \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha \left(2p^\mu - \gamma^\mu \not{p}\right)\not{p}'\left(2p^\beta - \not{p}\gamma^\beta\right)\gamma^\nu\right) \\
&= 4tr\left(\not{k}\not{p}\not{p}'\not{p}\right) - 2\eta_{\alpha\beta} tr\left(\not{k}\gamma^\alpha \not{p}'\not{p}\gamma^\beta \not{p}\right) \\
&\quad -2\eta_{\mu\nu} tr\left(\not{k}\not{p}\gamma^\mu \not{p}\not{p}'\gamma^\nu\right) + \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha\gamma^\mu \not{p}\not{p}'\not{p}\gamma^\beta\gamma^\nu\right) \\
&= 4tr\left(\not{k}\not{p}\left(2\left(p\cdot p'\right) - \not{p}\not{p}'\right)\right) - 2\eta_{\alpha\beta} tr\left(\not{k}\gamma^\alpha \not{p}'\left(2p^\beta - \gamma^\beta \not{p}\right)\not{p}\right) \\
&\quad -2\eta_{\mu\nu} tr\left(\not{k}\not{p}\left(2p^\mu - \not{p}\gamma^\mu\right)\not{p}'\gamma^\nu\right) + \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha\gamma^\mu\left(2\left(p\cdot p'\right) - \not{p}'\not{p}\right)\not{p}\gamma^\beta\gamma^\nu\right)
\end{aligned}$$

Simplifying, and using $p^2 = m^2 = 0$,

$$\begin{aligned}
T_p &= 8\left(p\cdot p'\right) tr\left(\not{k}\not{p}\right) - 4tr\left(\not{k}p^2\not{p}'\right) - 4tr\left(\not{k}\not{p}\not{p}'\not{p}\right) + 2\eta_{\alpha\beta} tr\left(\not{k}\gamma^\alpha \not{p}'\gamma^\beta p^2\right) \\
&\quad -4tr\left(\not{k}\not{p}\not{p}'\not{p}\right) + 2\eta_{\mu\nu} tr\left(\not{k}p^2\gamma^\mu \not{p}'\gamma^\nu\right) + \eta_{\alpha\beta}\eta_{\mu\nu} 2\left(p\cdot p'\right) tr\left(\not{k}\gamma^\alpha\gamma^\mu \not{p}\gamma^\beta\gamma^\nu\right) - \eta_{\alpha\beta}\eta_{\mu\nu} tr\left(\not{k}\gamma^\alpha\gamma^\mu \not{p}'\not{p}\not{p}\gamma^\beta\gamma^\nu\right) \\
&= 32\left(p\cdot p'\right)\left(k\cdot p\right) - 4tr\left(\not{k}\not{p}\not{p}'\not{p}\right) - 4tr\left(\not{k}\not{p}\not{p}'\not{p}\right)
\end{aligned}$$

$$
\begin{aligned}
& +2\left(p\cdot p'\right)\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^{\alpha}\gamma^{\mu}\not{p}\gamma^{\beta}\gamma^{\nu}\right) \\
= \quad & 32\left(p\cdot p'\right)(k\cdot p)-8tr\left(\not{k}\left(2\left(p\cdot p'\right)-\not{p}'\not{p}\right)\not{p}\right) \\
& +2\left(p\cdot p'\right)\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^{\alpha}\gamma^{\mu}\not{p}\gamma^{\beta}\gamma^{\nu}\right)
\end{aligned}
$$

Look at

$$
\begin{aligned}
\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^{\alpha}\gamma^{\mu}\not{p}\gamma^{\beta}\gamma^{\nu}\right) &= \eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\left(2k^{\alpha}-\gamma^{\alpha}\not{k}\right)\gamma^{\mu}\not{p}\left(2\eta^{\beta\nu}-\gamma^{\nu}\gamma^{\beta}\right)\right) \\
&= 4tr(\not{k}\not{p})-2\eta_{\mu\nu}tr(\gamma^{\mu}\not{p}\gamma^{\nu}\not{k}) \\
&\quad -2\eta_{\alpha\mu}tr(\gamma^{\alpha}\not{k}\gamma^{\mu}\not{p})+4\eta_{\mu\nu}tr(\not{k}\gamma^{\mu}\not{p}\gamma^{\nu}) \\
&= 4tr(\not{k}\not{p})-(2+2-4)\eta_{\mu\nu}tr(\gamma^{\mu}\not{p}\gamma^{\nu}\not{k}) \\
&= 16(k\cdot p)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T_p &= 32\left(p\cdot p'\right)(k\cdot p)-16\left(p\cdot p'\right)tr(\not{k}\not{p})+8tr\left(\not{k}\not{p}'\not{p}\not{p}\right)+32\left(p\cdot p'\right)(k\cdot p) \\
&= 32\left(p\cdot p'\right)(k\cdot p)-64\left(p\cdot p'\right)(p\cdot k)+32\left(p\cdot p'\right)(k\cdot p) \\
&= 0
\end{aligned}
$$

Now compute T_k, which is easier because the repeated ks are closer:

$$
\begin{aligned}
T_k &= \eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^{\alpha}\not{p}\gamma^{\mu}\not{p}'\gamma^{\beta}\not{k}\gamma^{\nu}\right) \\
&= \eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^{\alpha}\not{p}\gamma^{\mu}\not{p}'\gamma^{\beta}\left(2k^{\nu}-\gamma^{\nu}\not{k}\right)\right) \\
&= 2\eta_{\alpha\beta}tr\left(\not{k}\gamma^{\alpha}\not{p}\not{k}\not{p}'\gamma^{\beta}\right)-\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^{\alpha}\not{p}\gamma^{\mu}\not{p}'\gamma^{\beta}\gamma^{\nu}\not{k}\right) \\
&= 2\eta_{\alpha\beta}tr\left(\left(2k^{\alpha}-\gamma^{\alpha}\not{k}\right)\not{p}\not{k}\not{p}'\gamma^{\beta}\right) \\
&= 4tr\left(\not{p}\not{k}\not{p}'\not{k}\right)-2\eta_{\alpha\beta}tr\left(\gamma^{\alpha}\not{k}\not{p}\not{k}\not{p}'\gamma^{\beta}\right) \\
&= 4tr\left(\not{p}\not{k}\not{p}'\not{k}\right)-8tr\left(\not{k}\not{p}\not{k}\not{p}'\right) \\
&= -4tr\left(\not{p}\not{k}\left(2\left(p'\cdot k\right)-\not{k}\not{p}'\right)\right) \\
&= -8\left(p'\cdot k\right)tr(\not{p}\not{k})+4tr\left(\not{p}\not{k}\not{k}\not{p}'\right) \\
&= -32\left(p'\cdot k\right)(p\cdot k)
\end{aligned}
$$

and cycling $k\rightarrow p'\rightarrow p\rightarrow k$ we have

$$
T_{p'} = -32\left(p\cdot p'\right)\left(k\cdot p'\right)
$$

The full form of this cross term contribution to the squared matrix element it therefore,

$$
\begin{aligned}
\frac{1}{4}\sum_{all\,spins}\mathscr{M}^{s}_{fi}\mathscr{M}^{t*}_{fi} &= -\frac{e^4}{4st}\left(T_p+T_k-T_{p'}\right) \\
&= -\frac{e^4}{4st}\left(0-32\left(p'\cdot k\right)(p\cdot k)+32\left(p\cdot p'\right)\left(k\cdot p'\right)\right) \\
&= -\frac{8e^4}{st}\left(\left(p\cdot p'\right)\left(k\cdot p'\right)-\left(p'\cdot k\right)(p\cdot k)\right)
\end{aligned}
$$

1.5 Matrix squared: second cross term

The second cross term, averaged and summed over spins is

$$
\begin{aligned}
\frac{1}{4}\sum_{all\,spins}\mathcal{M}^t_{fi}\mathcal{M}^{s*}_{fi} &= \frac{1}{4}\sum_{all\,spins}\left(\frac{ie^2}{t}\eta_{\mu\nu}\bar{u}\left(p'\right)\gamma^\mu u\left(p\right)\bar{v}\left(k\right)\gamma^\nu v\left(k'\right)\right)\left(-\frac{ie^2}{s}\eta_{\alpha\beta}\bar{v}\left(k'\right)\gamma^\beta u\left(p'\right)\bar{u}\left(p\right)\gamma^\alpha v\left(k\right)\right) \\
&= \frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\sum_{all\,spins}\bar{u}_a\left(p'\right)\gamma^\mu_{ab}u_b\left(p\right)\bar{v}_c\left(k\right)\gamma^\nu_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\gamma^\beta_{ef}u_f\left(p'\right)\bar{u}_g\left(p\right)\gamma^\alpha_{gh}v_h\left(k\right) \\
&= (-1)^{15}\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\sum_{all\,spins}\gamma^\mu_{ab}u_b\left(p\right)\bar{u}_g\left(p\right)\gamma^\nu_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\gamma^\beta_{ef}u_f\left(p'\right)\bar{u}_a\left(p'\right)\gamma^\alpha_{gh}v_h\left(k\right)\bar{v}_c\left(k\right) \\
&= -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}\gamma^\mu_{ab}u_b\left(p\right)\bar{u}_g\left(p\right)\gamma^\alpha_{gh}v_h\left(k\right)\bar{v}_c\left(k\right)\gamma^\nu_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\gamma^\beta_{ef}u_f\left(p'\right)\bar{u}_a\left(p'\right) \\
&= -\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\gamma^\mu\not{p}\gamma^\alpha\not{k}\gamma^\nu\not{k}'\gamma^\beta\not{p}'\right)
\end{aligned}
$$

We need

$$
tr\left(\gamma^\mu\not{p}\gamma^\alpha\not{k}\gamma^\nu\not{k}'\gamma^\beta\not{p}'\right) = tr\left(\not{p}\gamma^\alpha\not{k}\gamma^\nu\not{k}'\gamma^\beta\not{p}'\gamma^\mu\right)
$$

Compare this to the first cross term, where we calculated

$$
\frac{e^4}{4st}\eta_{\alpha\beta}\eta_{\mu\nu}tr\left(\not{k}\gamma^\alpha\not{p}\gamma^\mu\not{p}'\gamma^\beta\not{k}'\gamma^\nu\right) = \frac{8e^4}{st}\left(\left(p\cdot p'\right)\left(k\cdot p'\right)-\left(p'\cdot k\right)\left(p\cdot k\right)\right)
$$

These differ only by the exchange $p\leftrightarrow k, p'\leftrightarrow k'$. Therefore,

$$
\frac{1}{4}\sum_{all\,spins}\mathcal{M}^t_{fi}\mathcal{M}^{s*}_{fi} = -\frac{8e^4}{st}\left(\left(k\cdot k'\right)\left(p\cdot k'\right)-\left(k'\cdot p\right)\left(k\cdot p\right)\right)
$$

1.6 Final matrix squared

The final matrix squared, averaged/summed over spins, is

$$
\begin{aligned}
\left|i\mathcal{M}_{fi}\right|^2 &= \left|i\mathcal{M}^s_{fi}\right|^2+\mathcal{M}^s_{fi}\mathcal{M}^{t*}_{fi}+\mathcal{M}^t_{fi}\mathcal{M}^{s*}_{fi}+\left|i\mathcal{M}^t_{fi}\right|^2 \\
&= \frac{8e^4}{s^2}\left(\left(p\cdot p'\right)\left(k\cdot k'\right)+\left(p'\cdot k\right)\left(p\cdot k'\right)+m^2\left(k'\cdot p'\right)+m^2\left(k\cdot p\right)+2m^4\right) \\
&\quad -\frac{8e^4}{st}\left(\left(p\cdot p'\right)\left(k\cdot p'\right)-\left(p'\cdot k\right)\left(p\cdot k\right)\right) \\
&\quad -\frac{8e^4}{st}\left(\left(k\cdot k'\right)\left(p\cdot k'\right)-\left(k'\cdot p\right)\left(k\cdot p\right)\right) \\
&\quad +\frac{8e^4}{t^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)+\left(p\cdot k\right)\left(p'\cdot k'\right)-m^2\left(p\cdot p'\right)-m^2\left(k\cdot k'\right)+2m^4\right)
\end{aligned}
$$

Dropping the mass terms and combining,

$$
\begin{aligned}
\left|i\mathcal{M}_{fi}\right|^2 &= \frac{8e^4}{s^2}\left(\left(p\cdot p'\right)\left(k\cdot k'\right)+\left(p'\cdot k\right)\left(p\cdot k'\right)\right)+\frac{8e^4}{t^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)+\left(p\cdot k\right)\left(p'\cdot k'\right)\right) \\
&\quad -\frac{8e^4}{st}\left(\left(p\cdot p'\right)\left(k\cdot p'\right)-\left(p'\cdot k\right)\left(p\cdot k\right)-\left(k'\cdot p\right)\left(k\cdot p\right)+\left(k\cdot k'\right)\left(p\cdot k'\right)\right)
\end{aligned}
$$

1.7 Relativistic kinematics

Consider the collision of an electron on a positron in the CM frame. Then the momenta are

$$
\begin{aligned}
p &= \left(E,\mathbf{p}\right) \\
k &= \left(E,-\mathbf{p}\right) \\
p' &= \left(E,\mathbf{p}'\right) \\
k' &= \left(E,-\mathbf{p}'\right)
\end{aligned}
$$

Then, dropping masses, and setting $\mathbf{p}^2 = E^2 - m^2 = E^2$, all quantities may be expressed in terms of E and θ:

$$\begin{aligned}
p\cdot k &= E^2+\mathbf{p}^2 \\
&= 2E^2-m^2 \\
&= 2E^2 \\
p'\cdot k &= E^2+\mathbf{p}\cdot\mathbf{p}' \\
&= E^2+\mathbf{p}^2\cos\theta \\
&= E^2(1+\cos\theta) \\
k'\cdot k &= E^2-\mathbf{p}\cdot\mathbf{p}' \\
&= E^2(1-\cos\theta) \\
p\cdot p' &= E^2(1-\cos\theta) \\
p\cdot k' &= E^2(1+\cos\theta) \\
p'\cdot k' &= E^2+\mathbf{p}'\cdot\mathbf{p}' \\
&= 2E^2 \\
t &= \left(p-p'\right)^2 \\
&= 2m^2-2p\cdot p' \\
&= -2E^2(1-\cos\theta) \\
s &= (p+k)^2 \\
&= 4E^2
\end{aligned}$$

Substituting into the squared matrix element,

$$\begin{aligned}
\frac{1}{4}\sum\left|i\mathscr{M}_{fi}\right|^2 &= \frac{8e^4}{s^2}\left(\left(p\cdot p'\right)\left(k\cdot k'\right)+\left(p'\cdot k\right)\left(p\cdot k'\right)\right)+\frac{8e^4}{t^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)+(p\cdot k)\left(p'\cdot k'\right)\right) \\
&\quad -\frac{8e^4}{st}\left(\left(p\cdot p'\right)\left(k\cdot p'\right)-\left(p'\cdot k\right)(p\cdot k)-\left(k'\cdot p\right)(k\cdot p)+\left(k\cdot k'\right)\left(p\cdot k'\right)\right) \\
&= \frac{8e^4}{4E^2 4E^2}\left(E^2(1-\cos\theta)E^2(1-\cos\theta)+E^2(1+\cos\theta)E^2(1+\cos\theta)\right)+\frac{8e^4}{2E^2(1-\cos\theta)2E^2(1-\cos\theta)}\left(E^2(1+\mathrm{co}\right. \\
&\quad -\frac{8e^4}{4E^2 t}\left(E^2(1-\cos\theta)E^2(1+\cos\theta)-E^2(1+\cos\theta)2E^2-E^2(1+\cos\theta)2E^2+E^2(1-\cos\theta)E^2(1+\cos\theta)\right)
\end{aligned}$$

Then the differential cross section is

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\left|\mathscr{M}_{ji}\right|^2 \\
&= \frac{e^4}{64\pi^2 4E^2}\left(\left(1+\cos^2\theta\right)+\frac{2\left(4+(1+\cos\theta)^2\right)}{(1-\cos\theta)^2}\right) \\
&\quad +\frac{e^4}{64\pi^2 4E^2}\left(\frac{1}{1-\cos\theta}\left(2\left(1-\cos^2\theta\right)-4(1+\cos\theta)\right)\right) \\
&= \frac{e^4}{64\pi^2 4E^2}\left(\left(1+\cos^2\theta\right)+\frac{2\left(4+(1+\cos\theta)^2\right)}{(1-\cos\theta)^2}+\frac{2\left(1-\cos^2\theta\right)}{1-\cos\theta}-\frac{4(1+\cos\theta)}{1-\cos\theta}\right) \\
&= \frac{e^4}{64\pi^2 2E^2}\left(\frac{1}{2}\left(1+\cos^2\theta\right)+\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}+2\cos^2\frac{\theta}{2}-\frac{2\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right) \\
&= \frac{e^4}{64\pi^2 2E^2}\left(\frac{1}{2}\left(1+\cos^2\theta\right)+\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}-\frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\right)
\end{aligned}$$

This is the ultrarelativistic limit of th Bhabha cross section (1936).

1.8 Traces of gamma matrices

Now compute the traces using the fundamental relation $\left\{\gamma^\alpha, \gamma^\beta\right\} = 2\eta^{\alpha\beta}$, and the cyclic property of the trace, $tr(A\ldots BC) = tr(CA\ldots B)$. First, we can show that the trace of the product of any odd number of γ-matrices vanishes by using $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^a\} = 0$,

$$
\begin{aligned}
tr\left(\underbrace{\gamma^\alpha \ldots \gamma^\beta}_{2n+1}\right) &= tr\left(1\gamma^\alpha \ldots \gamma^\beta\right) \\
&= tr\left(\gamma_5\gamma_5\gamma^\alpha \ldots \gamma^\beta\right) \\
&= -tr\left(\gamma_5\gamma^\alpha\gamma_5 \ldots \gamma^\beta\right) \\
&= (-1)^{2n+1} tr\left(\gamma_5\gamma^\alpha \ldots \gamma^\beta\gamma_5\right) \\
&= (-1)^{2n+1} tr\left(\gamma_5\gamma_5\gamma^\alpha \ldots \gamma^\beta\right) \\
&= -tr\left(\gamma^\alpha \ldots \gamma^\beta\right) \\
&= 0
\end{aligned}
$$

For even products, we will need traces of products of $2, 4, 6$ and 8 gamma matrices.

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\right) &= tr\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right) \\
&= -tr\left(\gamma^\beta\gamma^\alpha\right) + 2\eta^{\alpha\beta} tr(1) \\
&= -tr\left(\gamma^\alpha\gamma^\beta\right) + 8\eta^{\alpha\beta} \\
tr\left(\gamma^\alpha\gamma^\beta\right) &= 4\eta^{\alpha\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= tr\left(\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right)\gamma^\mu\gamma^\nu\right) \\
&= -tr\left(\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= -tr\left(\gamma^\beta(-\gamma^\mu\gamma^\alpha + 2\eta^{\mu\alpha})\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= tr\left(\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= tr\left(\gamma^\beta\gamma^\mu(-\gamma^\nu\gamma^\alpha) + 2\eta^{\nu\alpha}\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= -tr\left(\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha\right) + 2\eta^{\nu\alpha} tr\left(\gamma^\beta\gamma^\mu\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
2tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 2\eta^{\nu\alpha} tr\left(\gamma^\beta\gamma^\mu\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 4\eta^{\nu\alpha}\eta^{\beta\mu} - 4\eta^{\mu\alpha}\eta^{\beta\nu} + 4\eta^{\alpha\beta}\eta^{\mu\nu}
\end{aligned}
$$

For six, we use the simple pattern to more quickly find

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= tr\left(\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right)\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= tr\left(-\gamma^\beta(2\eta^{\alpha\mu} - \gamma^\mu\gamma^\alpha)\gamma^\nu\gamma^\rho\gamma^\sigma + 2\eta^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= tr\left(-\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha + 2\eta^{\alpha\sigma}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho - 2\eta^{\alpha\rho}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma + 2\eta^{\alpha\nu}\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\sigma - 2\eta^{\alpha\mu}\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma + 2\eta^{\alpha\beta}\gamma^\mu\gamma
\end{aligned}
$$

and from here we can use the result for the trace of four,

$$\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= tr\left(\eta^{\alpha\sigma}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho - \eta^{\alpha\rho}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma + \eta^{\alpha\nu}\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\sigma - \eta^{\alpha\mu}\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma + \eta^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= 4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\rho\beta}\eta^{\mu\nu}\right) - 4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma} - \eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\beta\sigma}\eta^{\mu\rho}\right) - 4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\nu\sigma} + \eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)
\end{aligned}$$

or, perhaps more mnemonically,

$$\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= 4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - 4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\nu\sigma} + \eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\beta\sigma}\eta^{\mu\rho}\right) - 4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma} - \eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\rho\beta}\eta^{\mu\nu}\right)
\end{aligned}$$

From this, if we don't run out of Greek letters, we can immediately write the result for eight gammas:

$$\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\lambda\gamma^\tau\right) &= 4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) - 4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\nu\sigma} + \eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\beta\sigma}\eta^{\mu\rho}\right) - 4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma} - \eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho} - \eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\rho\beta}\eta^{\mu\nu}\right)
\end{aligned}$$