1 Electron-positron annihilation into muon-antimuon pairs

1.1 The scattering matrix

From the Feynman diagram, we have the matrix element

$$i\mathscr{M}_{fi} = \bar{v}\left(p'\right)\left(-ie\gamma^{\alpha}\right)u\left(p\right)\frac{-i}{q^2}\left(\eta_{\alpha\beta}-\left(1-\xi\right)\frac{q_\alpha q_\beta}{q^2}\right)\bar{u}\left(k\right)\left(-ie\gamma^{\beta}\right)v\left(k'\right)$$

where p, p' are the incoming electron and positron momenta, respectively; $k, k'$are the outgoing muon and antimuon, respectively; $q = p + p' = k + k'$ is the s-channel 4-momentum; ξ allows a choice of gauge.

The gauge term drops out immediately, since

$$\begin{aligned}
\left(1-\xi\right)\frac{q_\alpha q_\beta}{q^2}\bar{u}\left(k\right)\left(-ie\gamma^{\alpha}\right)v\left(k'\right) &= \left(1-\xi\right)\frac{q_\alpha}{q^2}\bar{u}\left(k\right)\left(-ieq_\beta\gamma^{\beta}\right)v\left(k'\right) \\
&= -e\left(1-\xi\right)\frac{q_\alpha}{q^2}\bar{u}\left(k\right)\left(i\gamma^{\beta}k_\beta + i\gamma^{\beta}k'_\beta\right)v\left(k'\right) \\
&= -e\left(1-\xi\right)\frac{q_\alpha}{q^2}\left(\left(\bar{u}\left(k\right)i\gamma^{\beta}k_\beta\right)v\left(k'\right) + \bar{u}\left(k\right)\left(i\gamma^{\beta}k'_\beta v\left(k'\right)\right)\right) \\
&= -e\left(1-\xi\right)\frac{q_\alpha}{q^2}\left(-m\bar{u}\left(k\right)v\left(k'\right) + \bar{u}\left(k\right)mv\left(k'\right)\right) \\
&= 0
\end{aligned}$$

where we use the Dirac and conjugate Dirac equations in the penultimate step.

1.2 The sums over spins

Now set $q^2 = s$ and square the remaining matrix element,

$$\left|\mathscr{M}_{fi}\right|^2 = \frac{e^4}{s^2}\left(\bar{v}\left(p'\right)\gamma^{\alpha}u\left(p\right)\eta_{\alpha\beta}\bar{u}\left(k\right)\gamma^{\beta}v\left(k'\right)\right)\left(\bar{v}\left(k'\right)\gamma^{\mu}u\left(k\right)\eta_{\mu\nu}\bar{u}\left(p\right)\gamma^{\nu}v\left(p'\right)\right)$$

For unpolarized beams, and unmeasured final spins, we average over initial the two spins and sum over the two final spins,

$$\left(\frac{1}{4}\sum_{initial\,spins}\right)\left(\sum_{final\,spins}\right)\left|\mathscr{M}_{fi}\right|^2 = \frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}_{fi}\right|^2$$

This lets us use the outer product identities,

$$\begin{aligned}
\sum_{s=1}^{2}u\left(p\right)\bar{u}\left(p\right) &= \not{p}+m \\
\sum_{s=1}^{2}v\left(p\right)\bar{v}\left(p\right) &= \not{p}-m
\end{aligned}$$

To see what matrix products we end up with, write all spinor indices (all down, but still with the summation convention - the metric is δ_{ab}). Each time we move one spinor field past another, we introduce a sign, for a total of 14 exchanges in the second step.

$$\begin{aligned}
\frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}_{fi}\right|^2 &= \frac{e^4}{s^2}\frac{1}{4}\sum_{all\,spins}\left(\bar{v}_a\left(p'\right)\left[\gamma^{\alpha}\right]_{ab}u_b\left(p\right)\eta_{\alpha\beta}\bar{u}_c\left(k\right)\left[\gamma^{\beta}\right]_{cd}v_d\left(k'\right)\right)\left(\bar{v}_e\left(k'\right)\left[\gamma^{\mu}\right]_{ef}u_f\left(k\right)\eta_{\mu\nu}\bar{u}_g\left(p\right)\left[\gamma^{\nu}\right]_{gh}v_h\left(p'\right)\right) \\
&= \left(-1\right)^{14}\frac{e^4}{s^2}\frac{1}{4}\sum_{all\,spins}\left[\gamma^{\alpha}\right]_{ab}u_b\left(p\right)\bar{u}_g\left(p\right)\eta_{\alpha\beta}\left[\gamma^{\beta}\right]_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\left[\gamma^{\mu}\right]_{ef}u_f\left(k\right)\bar{u}_c\left(k\right)\eta_{\mu\nu}\left[\gamma^{\nu}\right]_{gh}v_h\left(p'\right)\bar{v}_a\left(p'\right) \\
&= \frac{e^4}{s^2}\frac{1}{4}\sum_{all\,spins}v_h\left(p'\right)\bar{v}_a\left(p'\right)\left[\gamma^{\alpha}\right]_{ab}u_b\left(p\right)\bar{u}_g\left(p\right)\eta_{\alpha\beta}u_f\left(k\right)\bar{u}_c\left(k\right)\left[\gamma^{\beta}\right]_{cd}v_d\left(k'\right)\bar{v}_e\left(k'\right)\left[\gamma^{\mu}\right]_{ef}\eta_{\mu\nu}\left[\gamma^{\nu}\right]_{gh}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^4}{s^2}\frac{1}{4}\left(\not{p}'-m_e\right)_{ha}[\gamma^\alpha]_{ab}(\not{p}+m_e)_{bg}\,\eta_{\alpha\beta}\left(\not{k}-m_\mu\right)_{fc}\left[\gamma^\beta\right]_{cd}\left(\not{k}'+m_\mu\right)_{de}[\gamma^\mu]_{ef}\,\eta_{\mu\nu}\,[\gamma^\nu]_{gh} \\
&= \frac{e^4}{s^2}\frac{1}{4}\eta_{\alpha\beta}\,\eta_{\mu\nu}\left(\left(\not{p}'-m_e\right)_{ha}[\gamma^\alpha]_{ab}(\not{p}+m_e)_{bg}[\gamma^\nu]_{gh}\right)\left(\left(\not{k}-m_\mu\right)_{fc}\left[\gamma^\beta\right]_{cd}\left(\not{k}'+m_\mu\right)_{de}[\gamma^\mu]_{ef}\right) \\
&= \frac{e^4}{s^2}\frac{1}{4}\eta_{\alpha\beta}\,\eta_{\mu\nu}tr\left(\left(\not{p}'-m_e\right)\gamma^\alpha(\not{p}+m_e)\gamma^\nu\right)tr\left(\left(\not{k}-m_\mu\right)\gamma^\beta\left(\not{k}'+m_\mu\right)\gamma^\mu\right)
\end{aligned}
$$

Notice that once we have identified the order of matrix products and the traces, we no longer need the spinor indices.

1.3 Traces of gamma matrices

Now compute the traces using the fundamental relation $\{\gamma^\alpha,\gamma^\beta\} = 2\eta^{\alpha\beta}$, and the cyclic property of the trace, $tr(A\ldots BC) = tr(CA\ldots B)$. First, we can show that the trace of the product of any odd number of γ-matrices vanishes by using $\gamma_5^2 = 1$ and $\{\gamma_5,\gamma^a\} = 0$,

$$
\begin{aligned}
tr\left(\underbrace{\gamma^\alpha\ldots\gamma^\beta}_{2n+1}\right) &= tr\left(1\gamma^\alpha\ldots\gamma^\beta\right) \\
&= tr\left(\gamma_5\gamma_5\gamma^\alpha\ldots\gamma^\beta\right) \\
&= -tr\left(\gamma_5\gamma^\alpha\gamma_5\ldots\gamma^\beta\right) \\
&= (-1)^{2n+1}tr\left(\gamma_5\gamma^\alpha\ldots\gamma^\beta\gamma_5\right) \\
&= (-1)^{2n+1}tr\left(\gamma_5\gamma_5\gamma^\alpha\ldots\gamma^\beta\right) \\
&= -tr\left(\gamma^\alpha\ldots\gamma^\beta\right) \\
&= 0
\end{aligned}
$$

For even products, we have

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\right) &= tr\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right) \\
&= -tr\left(\gamma^\beta\gamma^\alpha\right) + 2\eta^{\alpha\beta}tr(1) \\
&= -tr\left(\gamma^\alpha\gamma^\beta\right) + 8\eta^{\alpha\beta} \\
tr\left(\gamma^\alpha\gamma^\beta\right) &= 4\eta^{\alpha\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= tr\left(\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right)\gamma^\mu\gamma^\nu\right) \\
&= -tr\left(\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\nu\right) + 2\eta^{\alpha\beta}tr(\gamma^\mu\gamma^\nu) \\
&= -tr\left(\gamma^\beta(-\gamma^\mu\gamma^\alpha + 2\eta^{\mu\alpha})\gamma^\nu\right) + 2\eta^{\alpha\beta}tr(\gamma^\mu\gamma^\nu) \\
&= tr\left(\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu\right) - 2\eta^{\mu\alpha}tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta}tr(\gamma^\mu\gamma^\nu) \\
&= tr\left(\gamma^\beta\gamma^\mu(-\gamma^\nu\gamma^\alpha) + 2\eta^{\nu\alpha}\right) - 2\eta^{\mu\alpha}tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta}tr(\gamma^\mu\gamma^\nu) \\
&= -tr\left(\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha\right) + 2\eta^{\nu\alpha}tr\left(\gamma^\beta\gamma^\mu\right) - 2\eta^{\mu\alpha}tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta}tr(\gamma^\mu\gamma^\nu) \\
2tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 2\eta^{\nu\alpha}tr\left(\gamma^\beta\gamma^\mu\right) - 2\eta^{\mu\alpha}tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta}tr(\gamma^\mu\gamma^\nu) \\
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 4\eta^{\nu\alpha}\eta^{\beta\mu} - 4\eta^{\mu\alpha}\eta^{\beta\nu} + 4\eta^{\alpha\beta}\eta^{\mu\nu}
\end{aligned}
$$

Use these to evaluate the traces:

$$\begin{aligned}
tr\left(\left(\not{p}'-m_e\right)\gamma^\alpha\left(\not{p}+m_e\right)\gamma^\nu\right) &= tr\left(\not{p}'\gamma^\alpha\not{p}\gamma^\nu+\not{p}'\gamma^\alpha m_e\gamma^\nu-m_e\gamma^\alpha\not{p}\gamma^\nu-m_e^2\gamma^\alpha\gamma^\nu\right)\\
&= tr\left(\not{p}'\gamma^\alpha\not{p}\gamma^\nu\right)+tr\left(\not{p}'\gamma^\alpha m_e\gamma^\nu\right)-m_e tr\left(\gamma^\alpha\not{p}\gamma^\nu\right)-m_e^2 tr\left(\gamma^\alpha\gamma^\nu\right)\\
&= p'_\rho p_\sigma tr\left(\gamma^\rho\gamma^\alpha\gamma^\sigma\gamma^\nu\right)+m_e p'_\rho tr\left(\gamma^\rho\gamma^\alpha\gamma^\nu\right)-m_e p_\rho tr\left(\gamma^\alpha\gamma^\rho\gamma^\nu\right)-m_e^2 tr\left(\gamma^\alpha\gamma^\nu\right)\\
&= p'_\rho p_\sigma\left(4\eta^{\rho\alpha}\eta^{\sigma\nu}-4\eta^{\rho\sigma}\eta^{\alpha\nu}+4\eta^{\rho\nu}\eta^{\sigma\alpha}\right)-4m_e^2\eta^{\alpha\nu}\\
&= \left(4p'^\alpha p^\nu-4\left(p'\cdot p\right)\eta^{\alpha\nu}+4p'^\nu p^\alpha\right)-4m_e^2\eta^{\alpha\nu}
\end{aligned}$$

and, cycling the second trace

$$tr\left(\left(\not{k}-m_\mu\right)\gamma^\beta\left(\not{k}'+m_\mu\right)\gamma^\mu\right)=tr\left(\left(\not{k}'+m_\mu\right)\gamma^\mu\left(\not{k}-m_\mu\right)\gamma^\beta\right)$$

we see that simply replacing $p'\to k', p\to k$, the index change, $\alpha\to\beta, \nu\to\mu$, and interchanging the masses produces the right answer,

$$tr\left(\left(\not{k}-m_\mu\right)\gamma^\beta\left(\not{k}'+m_\mu\right)\gamma^\mu\right)=4k'^\beta k^\mu-4\left(k'\cdot k\right)\eta^{\beta\mu}+4k'^\mu k^\beta-4m_\mu^2\eta^{\beta\mu}$$

1.4 Lorentz contractions

With these traces, the matrix element becomes

$$\begin{aligned}
\frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}_{fi}\right|^2 &= \frac{e^4}{s^2}\frac{1}{4}\eta_{\alpha\beta}\eta_{\mu\nu}\left(4p'^\alpha p^\nu-4\left(p'\cdot p\right)\eta^{\alpha\nu}+4p'^\nu p^\alpha-4m_e^2\eta^{\alpha\nu}\right)\left(4k'^\beta k^\mu-4\left(k'\cdot k\right)\eta^{\beta\mu}+4k'^\mu k^\beta-4m_\mu^2\eta^{\beta\mu}\right)\\
&= \frac{e^4}{s^2}4\eta_{\alpha\beta}\eta_{\mu\nu}\left(p'^\alpha p^\nu-\left(p'\cdot p\right)\eta^{\alpha\nu}+p'^\nu p^\alpha-m_e^2\eta^{\alpha\nu}\right)\left(k'^\beta k^\mu-\left(k'\cdot k\right)\eta^{\beta\mu}+k'^\mu k^\beta-m_\mu^2\eta^{\beta\mu}\right)
\end{aligned}$$

Now perform the Lorentz contractions,

$$\begin{aligned}
\frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}_{fi}\right|^2 &= \frac{4e^4}{s^2}\eta_{\alpha\beta}\eta_{\mu\nu}\left(p'^\alpha p^\nu-\left(p'\cdot p\right)\eta^{\alpha\nu}+p'^\nu p^\alpha-m_e^2\eta^{\alpha\nu}\right)\left(k'^\beta k^\mu-\left(k'\cdot k\right)\eta^{\beta\mu}+k'^\mu k^\beta-m_\mu^2\eta^{\beta\mu}\right)\\
&= \frac{4e^4}{s^2}\left(\left(p\cdot k\right)\left(p'\cdot k'\right)-\left(k'\cdot k\right)\left(p'\cdot p\right)+\left(p'\cdot k\right)\left(p\cdot k'\right)-m_\mu^2\left(p'\cdot p\right)\right)\\
&\quad-\frac{4e^4}{s^2}\left(p'\cdot p\right)\left(\left(k'\cdot k\right)-4\left(k'\cdot k\right)+\left(k'\cdot k\right)-4m_\mu^2\right)\\
&\quad+\frac{4e^4}{s^2}\left(\left(p\cdot k'\right)\left(p'\cdot k\right)-\left(k'\cdot k\right)\left(p'\cdot p\right)+\left(p\cdot k\right)\left(p'\cdot k'\right)-m_\mu^2\left(p'\cdot p\right)\right)\\
&\quad-\frac{4e^4}{s^2}m_e^2\left(\left(k'\cdot k\right)-4\left(k'\cdot k\right)+\left(k'\cdot k\right)-4m_\mu^2\right)
\end{aligned}$$

Collecting terms,

$$\begin{aligned}
\frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}_{fi}\right|^2 &= \frac{8e^4}{s^2}\left(\left(p\cdot k\right)\left(p'\cdot k'\right)-\left(k'\cdot k\right)\left(p'\cdot p\right)+\left(p'\cdot k\right)\left(p\cdot k'\right)-m_\mu^2\left(p'\cdot p\right)\right)\\
&\quad+\frac{4e^4}{s^2}\left(\left(p'\cdot p\right)+m_e^2\right)\left(2\left(k'\cdot k\right)+4m_\mu^2\right)
\end{aligned}$$

1.5 Relativistic kinematics

Now choose the center of mass system. The initial and final particle pairs will each have equal but opposite 3-momenta,

$$\begin{aligned}
p &= \left(E,\mathbf{p}\right)\\
p' &= \left(E,-\mathbf{p}\right)\\
k &= \left(E,\mathbf{k}\right)\\
k' &= \left(E,-\mathbf{k}\right)
\end{aligned}$$

and s is related to the energy by

$$\begin{aligned} s &= \left(p+p'\right)^2 \\ &= \left((E,\mathbf{p})+(E,-\mathbf{p})\right)^2 \\ &= (2E,\mathbf{0})^2 \\ &= 4E^2 \end{aligned}$$

Let the angle between $\mathbf{p}$ and $\mathbf{k}$ be θ. We can eliminate $\mathbf{p}^2$ and $\mathbf{k}^2$ using

$$\begin{aligned} \mathbf{p}^2 &= E^2 - m_e^2 \\ &= \frac{s}{4} - m_e^2 \\ \mathbf{k}^2 &= \frac{s}{4} - m_\mu^2 \end{aligned}$$

We will need the six inner products,

$$\begin{aligned} p\cdot p' &= E^2 + \mathbf{p}^2 \\ &= \frac{s}{2} - m_e^2 \\ p\cdot k = p'\cdot k' &= E^2 - \mathbf{p}\cdot\mathbf{k} \\ &= E^2 - \sqrt{\frac{s}{4} - m_e^2}\sqrt{\frac{s}{4} - m_\mu^2}\cos\theta \\ &= \frac{s}{4}\left(1 - \sqrt{1-\frac{4m_e^2}{s}}\sqrt{1-\frac{4m_\mu^2}{s}}\cos\theta\right) \\ p\cdot k' = p'\cdot k &= \frac{s}{4}\left(1 + \sqrt{1-\frac{4m_e^2}{s}}\sqrt{1-\frac{4m_\mu^2}{s}}\cos\theta\right) \\ k\cdot k' &= E^2 + \mathbf{k}^2 \\ &= \frac{s}{2} - m_\mu^2 \end{aligned}$$

Now substitute these expressions into the matrix norm,

$$\begin{aligned} \frac{1}{4}\sum_{all\,spins}\left|\mathscr{M}_{fi}\right|^2 &= \frac{8e^4}{s^2}\left(\frac{s^2}{16}\left(1-\sqrt{1-\frac{4m_e^2}{s}}\sqrt{1-\frac{4m_\mu^2}{s}}\cos\theta\right)^2 - \left(\frac{s}{2}-m_\mu^2\right)\left(\frac{s}{2}-m_e^2\right) + \frac{s^2}{16}\left(1+\sqrt{1-\frac{4m_e^2}{s}}\sqrt{1-\frac{4m_\mu^2}{s}}\cos\theta\right)\right. \\ &\quad + \frac{4e^4}{s^2}\left(\left(\frac{s}{2}-m_e^2\right)+m_e^2\right)\left(2\left(\frac{s}{2}-m_\mu^2\right)+4m_\mu^2\right) \\ &= \frac{8e^4}{s^2}\left(\frac{s^2}{8}\left(1+\left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right) - \frac{s}{2}\frac{s}{2} + \frac{s}{2}m_e^2\right) \\ &\quad + \frac{8e^4}{s^2}\left(\frac{s}{2}s - m_e^2 s - 2m_\mu^2\frac{s}{2} + 2m_\mu^2 m_e^2 + 2\frac{s}{2}m_e^2 - 2m_e^2 m_\mu^2 + 4m_\mu^2\frac{s}{2} - 4m_\mu^2 m_e^2 + 4m_\mu^2 m_e^2\right) \\ &= e^4\left(\left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta - 1 + \frac{4m_e^2}{s}\right) + 4e^4\left(1+\frac{2m_\mu^2}{s}\right) \\ &= e^4\left(\left(1-\frac{4m_e^2}{s}\right)\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta + 1 + \frac{4}{s}\left(m_e^2+m_\mu^2\right)\right) \end{aligned}$$

1.6 Differential cross section

Now we can find the differential cross section. In the center of mass frame, the 2-particle to 2-particle differential cross section is

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\mathbf{k}|}{|\mathbf{p}|} \left|\mathscr{M}_{ji}\right|^2 d\Omega$$

where we have already carried out the integration of the remaining momentum space variables by using the conservation of momentum delta function. This immediately gives us the result,

$$d\sigma = \frac{e^4}{64\pi^2 s} \frac{\sqrt{1-\frac{4m_\mu^2}{s}}}{\sqrt{1-\frac{4m_e^2}{s}}} \left(1 + \frac{4}{s}\left(m_e^2 + m_\mu^2\right) + \left(1 - \frac{4m_e^2}{s}\right)\left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta\right) d\Omega$$

If the energy is much larger than either mass, $m_e < m_\mu \ll \sqrt{s} < 90\,GeV$, then this is approximately,

$$d\sigma = \frac{e^4}{64\pi^2 s}\left(1 + \cos^2\theta\right) d\Omega$$

Writing e^2 in terms of the fine structure constant,

$$e^2 = \frac{e^2}{\hbar c} = 4\pi\alpha$$

this is

$$d\sigma = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) d\Omega$$

Then the total cross section is

$$\begin{aligned}
\sigma &= \int d\sigma \\
&= \frac{\alpha^2}{4s}\int \left(1 + \cos^2\theta\right) d\cos\theta\, d\varphi \\
&= \frac{2\pi\alpha^2}{4s}\int_{-1}^{1}\left(1 + x^2\right) dx \\
&= \frac{2\pi\alpha^2}{4s}\left(x + \frac{1}{3}x^3\right)\Big|_{-1}^{1} \\
&= \frac{4\pi\alpha^2}{3s}
\end{aligned}$$