The Standard Model

April 29, 2010

Contents

1 Yang-Mills gauge theory

We begin with a discussion of Yang-Mills gauge theory for any special unitary group, $SU(n)$. We will develop a number of general properties of these theories. Then, in the next Section, we explore covariant derivatives in gauge theories of Lie groups. Then in Section 3, will begin our discussion of the standard model.

Begin the Lie group $\mathscr{P} \times SU(n)$, where $\mathscr{P} = I\mathscr{L}$ is the Poincaré (or Inhomogeneous Lorentz) group, $\mathscr{T}$ the Abelian group of translations, and $\mathscr{L} = Spin(1,3)$ is the Lorentz group. Form the group quotient

$$\mathscr{P} \times SU(n) / \mathscr{L} \times SU(n)$$

This gives an $\mathscr{L} \times SU(n)$ fiber bundle over a 4-dim base manifold. We modify this by generalizing the connection, requiring horizontality of the resulting curvatures/field strengths.

Notice that we have chosen the spin representation of the Lorentz generators. This means that, when we write an action with this symmetry, we can include spinor fields.

1.1 Generators for the Lie algebra

To carry this out explicitly, we require the Lie algebra of $\mathscr{P} \times SU(n)$. Since any unitary matrix may be written as

$$U = e^{iH}$$

where H is Hermitian, we may take arbitrary Hermitian matrices as generators. Since we require U to have unit determinant, we restrict our attention to traceless Hermitian matrices.

Exercise: By performing a unitary transformation to diagonalize H, prove the identity $\det U = e^{itr(H)}$, where U is unitary and H is Hermitian.

It is not difficult to write a basis for the Hermitian generators, which fall into three types. First, we have real, traceless, diagonal matrices, which are spanned by the set:

$$D_k = \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}$$

where the 1 occurs in the k^{th} row and column, where k runs from 1 to $n-1$. In components we may write

$$[D_k]^a{}_b = \delta^k_b \delta^a_k - \delta^{k+1}_b \delta^a_{k+1}$$

Next, consider the real, symmetric, off-diagonal matrices, with the general form

$$G^m{}_k = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & 1 & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & 1 & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

where the only nonzero elements occur in row k, column m and in row m, column k. In components,

$$[G^m{}_k]^a{}_b = \delta^m_b \delta^a_k + \delta_{kb}\delta^{ma}$$

If we wish to let k,m range freely over all values we need to subtract the trace,

$$[G^m{}_k]^a{}_b = \delta^m_b \delta^a_k + \delta_{kb}\delta^{ma} - 2\delta^m_k \delta^a_k \delta^m_b$$

Finally, we have pure imaginary, antisymmetric matrices of the form

$$\tilde{G}^m_{\ k} = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & -i & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & i & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

which may be written as

$$[G^m_{\ k}]^a_{\ b} = i\left(\delta^m_b \delta^a_k - \delta_{kb}\delta^{ma}\right)$$

Notice that $G^m_{\ k} = G^k_{\ m}$ and $\tilde{G}^m_{\ k} = -\tilde{G}^k_{\ m}$.

Exercise: Compute the commutation relations for this choice of the generators.

Exercise: Check that these satisfy

$$tr\left(\lambda_a \lambda_b\right) = 2\delta_{ab}$$

Now recall that the special unitary matrices have *anti*-Hermitian generators. If we want to describe the *real* group manifold using the structure constants, we need real structure constants, and these arise from the commutation relations of the anti-Hermitian form of the generators. However, we have found $f_{AB}^{\ \ C}$ for the Hermitian form,

$$[G_A, G_B] = f_{AB}^{\ \ C} G_C$$

The anti-Hermitian generators are iG_A, so the correct structure constants are given by multiplying both sides by i^2,

$$[iG_A, iG_B] = if_{AB}^{\ \ C}\left(iG_C\right)$$

The real-valued structure constants are therefore $c_{AB}^{\ \ C} = if_{AB}^{\ \ C}$.

For the Poincaré group we have seen before that we may write

$$\begin{aligned} [M^a_b, M^c_d] &= -\frac{1}{2}\left(M^a_{\ d}\eta_{bc} - M^a_{\ c}\eta_{bd} - M_{bd}\delta^a_c + M_{bc}\delta^a_d\right) \\ [M^a_b, P_c] &= -\frac{1}{2}\left(\delta^a_c P_b - \eta_{bc}\eta^{ad}P_d\right) \\ [P_a, P_b] &= 0 \end{aligned}$$

Since the groups $\mathscr{P}$ and $SU(3)$ are in a direct product, the commutators between the generators from any pair of these different subgroups commute,

$$[G_A, P_a] = 0 = [G_A, M^a_b]$$

This completes the Lie algebra.

1.2 Maurer-Cartan structure equations

From the Lie algebra, we may construct the Maurer-Cartan equations. For each generator we introduce dual 1-forms,

$$\begin{aligned} \left\langle G_A, \omega^B \right\rangle &= \delta^B_A \\ \left\langle M^a_b, \omega^c_d \right\rangle &= \delta^a_d \delta^c_b - \eta^{ac}\eta_{bd} \\ \left\langle P_a, \mathbf{e}^b \right\rangle &= \delta^b_a \end{aligned}$$

The Maurer-Cartan equations are immediate,

$$d\omega^\Delta = -\frac{1}{2} c_{\Lambda\Sigma}^{\ \ \ \Delta} \omega^\Lambda \wedge \omega^\Sigma$$

Substituting the different generators:

$$\begin{aligned}
\mathbf{d}\omega^{\alpha}_{\ \beta} &= \omega^{\mu}_{\ \beta}\wedge\omega^{\alpha}_{\ \mu} \\
\mathbf{d}\mathbf{e}^{\alpha} &= \mathbf{e}^{\beta}\wedge\omega^{\alpha}_{\ \beta} \\
\mathbf{d}\omega^{A} &= -\frac{i}{2}f_{BC}{}^{A}\omega^{B}\wedge\omega^{C}
\end{aligned}$$

1.3 Modifying the connections

Now, we generalize the connection 1-forms. This means that the Maurer-Cartan equations will pick up extra, tensorial 2-form terms. These are the field strengths; we require them to be horizontal forms, that is, they expand in the forms $\mathbf{e}^{\alpha}$ spanning the base manifold only. We will not generalize the Lorentz or translational generators, because we do not yet know how to quantize gravity, and it is difficult to quantize the remaining fields if the background spacetime is curved. Requiring the spacetime curvature and the torsion to vanish means that the solder form and spin connection are still described by the Maurer-Cartan equations,

$$\begin{aligned}
\mathbf{d}\omega^{\alpha}_{\ \beta} &= \omega^{\mu}_{\ \beta}\wedge\omega^{\alpha}_{\ \mu} \\
\mathbf{d}\mathbf{e}^{\alpha} &= \mathbf{e}^{\beta}\wedge\omega^{\alpha}_{\ \beta}
\end{aligned}$$

The first of these shows that the spin connection is pure gauge,

$$\omega^{\alpha}_{\ \beta} = -\mathbf{d}\Lambda^{\alpha}_{\ \mu}\bar{\Lambda}^{\mu}_{\ \beta}$$

where $\Lambda^{\alpha}_{\ \beta}$ is a local Lorentz transformation, and $\bar{\Lambda}^{\alpha}_{\ \beta}$ its inverse. These transformations provide coordinates on the Lorentz part of the fiber bundle, and we may choose them constant. Then the spin connection vanishes and the equation for the solder form reduces to

$$\mathbf{d}\mathbf{e}^{\alpha} = 0$$

The solder form is thus exact. We cannot choose these forms to be zero since they must span the base manifold, but their being exact allows us to write them as differentials of coordinate functions,

$$\mathbf{e}^{\alpha} = \mathbf{d}x^{\alpha}$$

The x^{α} are the usual Cartesian coordinates on Minkowski space. If we wanted to use, say, spherical coordinates, then the spin connection would still be pure gauge, but not zero. Since these span the base manifold, all of the (horizontal) curvatures/field strengths must be expanded in terms of them only.

The remaining Maurer-Cartan equation now generalize to

$$\mathbf{d}\omega^{A} = -\frac{i}{2}f_{BC}{}^{A}\omega^{B}\wedge\omega^{C}+\mathbf{F}^{A}$$

where $\mathbf{F}^{A}=\frac{1}{2}F^{A}_{\alpha\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta}$. This equation gives the expression for the field strength. Expanding the differential forms, we have

$$\frac{1}{2}F^{A}_{\alpha\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta} = \partial_{\alpha}\omega^{A}_{\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta}+\frac{1}{2}f_{BC}{}^{A}\omega^{B}_{\alpha}\omega^{C}_{\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta}$$

where $F^{A}_{\alpha\beta}=-F^{A}_{\beta\alpha}$. Antisymmetrizing the right side and dropping the basis forms gives the coordinate expression for the field strength:

$$F^{A}_{\alpha\beta}=\partial_{\alpha}\omega^{A}_{\beta}-\partial_{\beta}\omega^{A}_{\alpha}+f_{BC}{}^{A}\omega^{B}_{\alpha}\omega^{C}_{\beta}$$

Notice that for an Abelian group, $f_{BC}{}^{A}=0$, and this reduces to the same form as the Maxwell field tensor. The extra quadratic terms gives the weak and strong interactions considerable richness absent from electromagnetism. Notice also that there is one field for each of the n^2-1 generators of $SU(3)$.

2 Covariant derivatives in gauge theories of Lie groups

The connection forms for the color and electroweak symmetries allow us to define two covariant derivatives. We begin by describing the general case of a group-covariant derivative.

The results in this Section apply to any simple Lie group, so we change the notation slightly. We will work with an arbitrary linear representation with generators G_A and real structure constants $c_{AB}{}^{C}$.

Suppose we have a linear representation of a simple Lie non-Abelian group $\mathscr{G}$ with generators G_A and Lie algebra

$$[G_A, G_B] = c_{AB}{}^{C} G_C$$

A linear representation of a Lie group is defined by specifying a vector space, $\mathscr{V}$, on which the group acts. The group elements may then be written as matrices.

Let $\mathbf{h}^A$ be 1-forms dual to the generators,

$$\left\langle G_A, \mathbf{h}^B \right\rangle = \delta_A^B$$

and define

$$\mathbf{h} = \mathbf{h}^A G_A$$

For functions over the group manifold, we simply define $\mathbf{D}f = \mathbf{d}f$. Now let $v \in \mathscr{V} \times \mathscr{G}$ be a vector field over the group manifold. Then the covariant derivative of v is defined as

$$\mathbf{D}v = \mathbf{d}v + \mathbf{h}v$$

where v and the connection transform under the action of $g \in \mathscr{G}$, according to

$$\begin{aligned} v' &= gv \\ \mathbf{h}' &= g\mathbf{h}g^{-1} - \mathbf{d}g\, g^{-1} \end{aligned}$$

We define the covariant derivative on higher rank tensors by requiring $\mathbf{D}$ to satisfy the Leibnitz property.

We have three things to prove:

1. To be a derivation, $\mathbf{D}$ must be linear and Leibnitz. We have made it Leibnitz by definition, but need to show linearity.

2. We must show that the Leibnitz property uniquely defines $\mathbf{D}$ on any rank of tensor.

3. We must prove that $\mathbf{D}$ is covariant with respect to the action of $\mathscr{G}$.

Linearity is immediate. Let $w = au + bv$, where $u, v, w \in \mathscr{V}$ and a, b are real numbers.

$$\begin{aligned} \mathbf{D}w &= \mathbf{d}w + \mathbf{h}w \\ &= \mathbf{d}(au + bv) + \mathbf{h}(au + bv) \\ &= a\mathbf{d}u + b\mathbf{d}v + a\mathbf{h}u + b\mathbf{h}v \\ &= a\mathbf{D}u + b\mathbf{D}v \end{aligned}$$

For the Leibnitz property, we need the additivity of tensorial ranks. If we denote the outer product of $u, v \in \mathscr{V}$ by $w = uv$, then the rank of w is the sum of the ranks of u and v (in this case, we specified that u and v are in $\mathscr{V}$ and therefore rank 1, so w is rank 2, but we can then iterate the procedure to arbitrary higher ranks). It is easiest to see what is happening here if we include the matrix indices. Requiring the Leibnitz product rule for the outer product of two vectors, we have

$$\mathbf{D}\left(u^A v^B\right) = \left(\mathbf{D}u^A\right) v^B + u^A \mathbf{D}v^B$$

Rewriting this in terms of $w^{AB} = u^A v^B$, we have

$$\begin{aligned}
\mathbf{D}w^{AB} &= \mathbf{D}\left(u^A v^B\right) \\
&= \left(\mathbf{D}u^A\right) v^B + u^A \mathbf{D}v^B \\
&= \left(\mathbf{d}u^A + \mathbf{h}^A_{\ C} u^C\right) v^B + u^A \left(\mathbf{d}v^B + \mathbf{h}^B_{\ C} v^C\right) \\
&= \left(\mathbf{d}u^A\right) v^B + u^A \mathbf{d}v^B + \mathbf{h}^A_{\ C} u^C v^B + \mathbf{h}^B_{\ C} u^A v^C \\
&= \mathbf{d}\left(u^A v^B\right) + \mathbf{h}^A_{\ C} u^C v^B + \mathbf{h}^B_{\ C} u^A v^C \\
&= \mathbf{d}w^{AB} + \mathbf{h}^A_{\ C} w^{CB} + \mathbf{h}^B_{\ C} w^{AC}
\end{aligned}$$

which is the usual expression for a covariant derivative. Since any second rank tensor may be written as a linear combination of outer products of vectors, the linearity of the derivative implies this form for the covariant derivative of arbitrary second rank tensors.

Exercise Generalize this result to products of tensors of arbitrary rank.

Finally, we turn to covariance. We have

$$\begin{aligned}
\mathbf{D}'v' &= \mathbf{d}v' + \mathbf{h}'v' \\
&= \mathbf{d}(gv) + \left(g\mathbf{h}g^{-1} - \mathbf{d}g\, g^{-1}\right) gv \\
&= \mathbf{d}g\, v + g\mathbf{d}v + g\mathbf{h}v - \mathbf{d}g\, v \\
&= g\left(\mathbf{d}v + \mathbf{h}v\right) \\
&= g\mathbf{D}v
\end{aligned}$$

Exercise Rewrite this derivation, putting in all of the indices.

We will also need the special case of an Abelian group, since $U(1)$ is Abelian. Here the notion of weight (or charge) replaces the rank, and even functions may have arbitrary weight. Under the action of $\mathscr{G}$, any field (scalar, vector, tensor) of weight w transforms as

$$\phi' = g^w \phi$$

that is, g acts w times on ϕ. Then the covariant derivative is defined as

$$\mathbf{D}\phi = \mathbf{d}\phi + w\mathbf{h}\phi$$

Linearity and the Leibnitz property are immediate, provided we only perform sums of fields of equal weight, and define the weight of a product of two fields to be the sum of the weights of the two fields separately. Then:

$$\begin{aligned}
\mathbf{D}(a\phi + b\chi) &= \mathbf{d}(a\phi + b\chi) + w\mathbf{h}(a\phi + b\chi) \\
&= a\mathbf{d}\phi + b\mathbf{d}\chi + aw\mathbf{h}\phi + bw\mathbf{h}\chi \\
&= a\mathbf{D}\phi + b\mathbf{D}\chi
\end{aligned}$$

and for fields ϕ, ψ of weights w_ϕ, w_ψ we have

$$\begin{aligned}
\mathbf{D}(\phi\psi) &= \mathbf{d}(\phi\psi) + w_{\phi\psi}\mathbf{h}\phi\psi \\
&= (\mathbf{d}\phi)\,\psi + \phi\mathbf{d}\psi + \left(w_\phi + w_\psi\right)\mathbf{h}\phi\psi \\
&= \left(\mathbf{d}\phi + w_\phi\mathbf{h}\phi\right)\psi + \phi\left(\mathbf{d}\psi + w_\psi\mathbf{h}\psi\right) \\
&= (\mathbf{D}\phi)\,\psi + \phi\mathbf{D}\psi
\end{aligned}$$

Now consider covariance. For an Abelian group, the transformation of the connection by a group element $g = e^{\beta^A G_A}$ reduces to

$$\begin{aligned}
\mathbf{h}' &= g\mathbf{h}g^{-1} - \mathbf{d}g\, g^{-1} \\
&= \mathbf{h} - \mathbf{d}\left(e^{\beta^A G_A}\right) e^{-\beta^A G_A} \\
&= \mathbf{h} - \mathbf{d}\beta^A G_A
\end{aligned}$$

Checking covariance, we have

$$\begin{aligned}
\mathbf{D}'\phi' &= \mathbf{d}\phi' + w\mathbf{h}'\phi' \\
&= \mathbf{d}\left(g^{w}\phi\right) + w\left(\mathbf{h} - \mathbf{d}\beta^{A}G_{A}\right)g^{w}\phi \\
&= \mathbf{d}\left(e^{w\beta^{A}G_{A}}\phi\right) + w\mathbf{h}g^{w}\phi - w\left(\mathbf{d}\beta^{A}\right)G_{A}g^{w}\phi \\
&= e^{w\beta^{A}G_{A}}\mathbf{d}\phi + w\left(\mathbf{d}\beta^{A}\right)G_{A}e^{w\beta^{A}G_{A}}\phi + w\mathbf{h}g^{w}\phi - w\left(\mathbf{d}\beta^{A}\right)G_{A}g^{w}\phi \\
&= g^{w}\mathbf{d}\phi + w\left(\mathbf{d}\beta^{A}\right)G_{A}g^{w}\phi + w\mathbf{h}g^{w}\phi - w\left(\mathbf{d}\beta^{A}\right)G_{A}g^{w}\phi \\
&= g^{w}\left(\mathbf{d}\phi + w\mathbf{h}\phi\right) \\
&= g^{w}\mathbf{D}\phi
\end{aligned}$$

so that the covariant derivative of the field transforms with the same weight as the field.

Finally, suppose the Lie group is a direct product of Lie groups, $\mathscr{G} = \mathscr{G}_1 \times \mathscr{G}_2$. The the Lie algebra is the direct sum of the Lie algebras of $\mathscr{G}_1$ and $\mathscr{G}_2$. The generators involved in the connection commute, so the covariant derivative is found by just adding the two connections,

$$\mathbf{D}v = \mathbf{d}v + \mathbf{h}_1 v + \mathbf{h}_2 v$$

We conclude the section by expanding these results into components. For generators G_A and gauge 1-forms $\mathbf{h}^A = h^A_{\ \alpha}\mathbf{d}x^{\alpha}$, we expand the forms to find the component expression,

$$D_{\alpha}v^{a} = \partial_{\alpha}v^{a} + h^{A}_{\ \alpha}\left[G_{A}\right]^{a}_{\ \ b}v^{b}$$

For unitary groups, it is convenient to write the generators as $G_A = \frac{i}{2}gH_A$, where H_A is Hermitian and the factor $\frac{g}{2}$ introduces a coupling constant g with a conventional factor of $\frac{1}{2}$. Ultimately, g is the unit of charge of the field – electric charge, color charge, weak hypercharge, etc. Then the covariant derivative becomes

$$D_{\alpha}v^{a} = \partial_{\alpha}v^{a} + \frac{i}{2}gh^{A}_{\ \alpha}\left[H_{A}\right]^{a}_{\ \ b}v^{b}$$

The gauge transformations

$$\begin{aligned}
\tilde{\mathbf{h}} &= g\mathbf{h}g^{-1} - \mathbf{d}g\,g^{-1} \\
\tilde{h}^{A}_{\beta}G_{A} &= h^{A}_{\beta}\left(gG_{A}g^{-1}\right) - \left(\partial_{\beta}g\,g^{-1}\right)^{A}G_{A}
\end{aligned}$$

are not simple, but for an infinitesimal gauge transformation we have $g \approx 1 + \varepsilon^{A}G_{A}$

$$\begin{aligned}
\delta h^{A}_{\beta}G_{A} &= h^{A}_{\beta}\left(1 + \varepsilon^{B}G_{B}\right)G_{A}\left(1 - \varepsilon^{C}G_{C}\right) - \partial_{\beta}\varepsilon^{A}G_{A} - h^{A}_{\beta}G_{A} \\
&= h^{A}_{\beta}\varepsilon^{B}\left[G_{B}, G_{A}\right] - \partial_{\beta}\varepsilon^{A}G_{A} \\
&= c_{BC}^{\ \ \ A}\varepsilon^{B}h^{C}_{\beta}G_{A} - \partial_{\beta}\varepsilon^{A}G_{A} \\
\delta h^{A}_{\beta} &= c_{BC}^{\ \ \ A}\varepsilon^{B}h^{C}_{\beta} - \partial_{\beta}\varepsilon^{A}
\end{aligned}$$

3 Yang-Mills action and conserved currents

Once we have the field strengths and curvatures of the gauge theory, we can identify the relevant tensor fields, and write an action. From the action we can find the field equations and any conserved quantities.

3.1 Group tensors

The action may be constructed from any of the spacetime tensors

$$\eta_{\alpha\beta}, \delta^{\alpha}_{\beta}, e_{\alpha\beta\mu\nu}$$

and any of the $SU(n)$-invariant tensors,

$$K_{AB} = tr(G_A G_B), \delta_{ab}, F^A_{\alpha\beta}$$

We may also use any representations of the gauge group, that is, tensors under $Spin(3,1)$ and $SU(n)$. The basic vector spaces that define our representations tell us what we can use. Since we chose the spinor representation of the Lorentz group, we may use Dirac spinors, ψ, and any tensors constructible from them. Our representation of $SU(n)$ allows us to use any complex, n-dim vectors and the tensors built from them.

Of particular interest are the bispinor combinations involving the Dirac matrices. From any spinor, ψ, and its conjugate, $\bar{\psi} = \psi^\dagger \gamma^0$, we have the Lorentz covariant combinations,

$$\begin{array}{rl} \bar{\psi}\psi & scalar \\ \bar{\psi}\gamma^\alpha\psi & vector \\ \bar{\psi}\sigma^{\alpha\beta}\psi & tensor \\ \bar{\psi}\gamma^\alpha\gamma_5\psi & pseudovector \\ \bar{\psi}\gamma_5\psi & pseudoscalar \end{array}$$

We may also take higher order combinations,

$$\psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_k$$

though these are rarely needed or used.

For $SU(n)$, we may use multiplets of any of the Lorentz covariant objects. Each of the quantities above transforms as a scalar under $SU(n)$, but we can make n-tuples of any of them. The simplest cases, and the ones we will use, are n complex scalars,

$$\phi^a$$

and n spinor fields,

$$\psi^a$$

called scalar and spinor multiplets, respectively. But we could also take, say, n tensor fields

$$f_{ab}{}^c \bar{\psi}^a \sigma^{\alpha\beta} \psi^b$$

or an $SU(n)$ tensor like

$$T^{ab\alpha} = \bar{\psi}^a \gamma^\alpha \gamma_5 \psi^b$$

This object transforms as a pseudovector under Lorentz transformations and as a second rank tensor under $SU(n)$.

We will restrict our attention to the simplest possibilities.

3.2 The Yang-Mills action

A typical Yang-Mills action consists of a kinetic term for the gauge fields, built quadratically from the field strength and the $SU(n)$ Killing metric $K_{AB} = \delta_{AB}$, and an n-tuple of spinor fields. The two are coupled through the covariant derivative. The action is then

$$\begin{aligned} S &= \int \frac{1}{4} K_{AB} \eta^{\alpha\mu} \eta^{\beta\nu} F^A_{\alpha\beta} F^B_{\mu\nu} + \delta_{ab} \bar{\psi}^a (i\not{D} - m) \psi^b \\ &= \int \frac{1}{4} F^A_{\alpha\beta} F^{A\alpha\beta} + \bar{\psi}^a (i\not{D} - m) \psi^a \end{aligned}$$

Historically, these gauge theories were built the other way around. One started with, say, the Dirac action for a multiplet, $\int \bar{\psi}^a (i\not{\partial} - m) \psi^a$. Then noting that it has a global $SU(n)$ symmetry, a systematic extension to a locally $SU(n)$ invariant theory leads to S above. The present approach is better suited to gauge theories which include gravity, since gravitational gauge theory requires the construction of the base manifold.

3.3 Conserved currents

Now that we have an action invariant under $\mathscr{P} \times SU(n)$, we can apply Noether's theorem to find conserved quantities. The spacetime symmetries lead to the usual energy-momentum tensors,

$$\begin{aligned} T_{\alpha\beta} &= \eta^{\mu\nu} F^A_{\alpha\mu} F^A_{\beta\nu} - \frac{1}{4}\eta_{\alpha\beta} F^A_{\mu\nu} F^{A\mu\nu} \\ S_{\alpha\beta} &= \frac{i}{2} g_{\mu\beta} \bar{\psi} e_b{}^{\mu} \gamma^b \overleftrightarrow{D}_\alpha \psi - \frac{1}{4} g_{\alpha\beta} \left(i\bar{\psi}\gamma^b e_b{}^{\beta} D_\beta \psi - m\bar{\psi}\psi \right) \end{aligned}$$

where the second applies to a single spinor field. For a multiplet, we simply sum over all spinors in the multiplet.

Exercise: Derive these expressions for the energy-momentum tensors by first writing the action using general coordinates (so that $\eta_{\alpha\beta}$ is replaced by a general metric, $g_{\alpha\beta}$, and the volume form d^4x is replaced by $\sqrt{-g}d^4x$), then varying the metric. For the spinor case, the action may be written using the solder form instead

$$S = \int d^4x\, e \left(i\bar{\psi}\gamma^a e_a{}^{\alpha} D_\alpha \psi - m\bar{\psi}\psi \right)$$

where $e = \sqrt{-g}$, and correcting the indices at the end.

We write the action in the symmetric form,

$$S = \int \frac{1}{4} F^A_{\alpha\beta} F^{A\alpha\beta} + \frac{i}{2}\bar{\psi}^a\gamma^\alpha \left(D_\alpha \psi^a\right) - \frac{i}{2}\left(D_\alpha \bar{\psi}^a\right)\gamma^\alpha \psi^a - m\bar{\psi}^a\psi^a$$

For $SU(n)$, a general $SU(n)$ variation of the action gives

$$\begin{aligned} \delta S &= \int \frac{1}{2}\delta F^A_{\alpha\beta} F^{\alpha\beta}_A + \frac{i}{2}\delta\bar{\psi}^a\gamma^\alpha \left(D_\alpha\psi^a\right) + \frac{i}{2}\bar{\psi}^a\gamma^\alpha\left(\delta D_\alpha \psi^a\right) + \frac{i}{2}\bar{\psi}^a\gamma^\alpha\left(D_\alpha\delta\psi^a\right) \\ &\quad - \frac{i}{2}\left(\delta D_\alpha\bar{\psi}^a\right)\gamma^\alpha\psi^a - \frac{i}{2}\left(D_\alpha\delta\bar{\psi}^a\right)\gamma^\alpha\psi^a - \frac{i}{2}\left(D_\alpha\bar{\psi}^a\right)\gamma^\alpha\delta\psi^a - m\delta\bar{\psi}^a\psi^a - m\bar{\psi}^a\delta\psi^a \\ &= \int D_\alpha \delta B^A_\beta F^{\alpha\beta}_A + \frac{i}{2}\delta\bar{\psi}^a\gamma^\alpha\left(D_\alpha\psi^a\right) + \frac{i}{2}\bar{\psi}\gamma^\beta\left(\delta B^A_\beta G_A \psi\right) \\ &\quad + \frac{i}{2} D_\alpha\left(\bar{\psi}^a\gamma^\alpha\delta\psi^a\right) - \frac{i}{2}D_\alpha\left(\bar{\psi}^a\gamma^\alpha\right)\delta\psi^a \\ &\quad - \frac{i}{2}\delta B^A_\beta G_A \bar{\psi}^a\gamma^\beta\psi^a - \frac{i}{2}D_\alpha\left(\delta\bar{\psi}^a\gamma^\alpha\psi^a\right) + \frac{i}{2}\delta\bar{\psi}^a\gamma^\alpha D_\alpha\psi^a \\ &\quad - \frac{i}{2}\left(D_\alpha\bar{\psi}^a\right)\gamma^\alpha\delta\psi^a - m\delta\bar{\psi}^a\psi^a - m\bar{\psi}^a\delta\psi^a \end{aligned}$$

which finally rearranges into surface terms and field equation terms:

$$\begin{aligned} \delta S &= \int D_\alpha\left(\delta B^A_\beta F^{\alpha\beta}_A + \frac{i}{2}\bar{\psi}^a\gamma^\alpha\delta\psi^a - \frac{i}{2}\delta\bar{\psi}^a\gamma^\alpha\psi^a\right) \\ &\quad - \delta B^A_\beta\left(D_\alpha F^{\alpha\beta}_A - \frac{i}{2}\bar{\psi}\gamma^\beta G_A\psi + \frac{i}{2}G_A\bar{\psi}^a\gamma^\beta\psi^a\right) \\ &\quad + \delta\bar{\psi}^a\left(i\not{D}\psi^a - m\psi^a\right) \\ &\quad - \left(i\bar{\psi}^a\overleftarrow{\not{D}}\gamma^\alpha + m\bar{\psi}^a\right)\delta\psi^a \end{aligned}$$

Now impose the field equations,

$$\begin{aligned} D_\alpha F^{\alpha\beta}_A &= \frac{i}{2}\delta_{ab}\bar{\psi}^a\gamma^\beta\left[G_A\right]^b{}_c\psi^c - \frac{i}{2}\delta_{ab}\left[G_A\right]^a{}_c\bar{\psi}^c\gamma^\beta\psi^b \\ &= i\bar{\psi}\gamma^\beta G_A\psi^c \\ i\not{D}\psi^a - m\psi^a &= 0 \\ i\bar{\psi}^a\overleftarrow{\not{D}}\gamma^\alpha + m\bar{\psi}^a &= 0 \end{aligned}$$

and restrict the variation to a global gauge change,

$$
\begin{aligned}
\delta B^A_\beta &= -\left(\partial_\beta \varepsilon^A + c_{BC}{}^A B^B_\beta \varepsilon^C\right) \\
&= -D_\beta \varepsilon^A \\
&= 0 \\
\delta\psi^a &= i\varepsilon^A (G_A\psi)^a \\
\delta\bar{\psi}^a &= -i\varepsilon^A (\bar{\psi} G_A)^a
\end{aligned}
$$

Then the variation, δS, vanishes identically, and we are left with the conserved current,

$$
\begin{aligned}
J^\beta &= -\left(\delta B^A_\beta F^{\alpha\beta}_A + \frac{i}{2}\bar{\psi}^a\gamma^\alpha\delta\psi^a - \frac{i}{2}\delta\bar{\psi}^a\gamma^\alpha\psi^a\right) \\
&= \varepsilon^A\bar{\psi}\gamma^\alpha G_A\psi
\end{aligned}
$$

Since ε^A is arbitrary, we get one conserved current for each generator,

$$
J^\beta_A = \bar{\psi}\gamma^\alpha G_A\psi
$$

and this is exactly the collection of source currents for the Yang-Mills field.

This concludes our general remarks on Yang-Mills theories and covariant derivatives. We now turn to the Standard Model.

4 SU(3) x SU(2) x U(1) gauge theory

Begin the Lie group $\mathscr{P}\times SU(3)\times SU(2)\times U(1)$, where $\mathscr{P}=\mathscr{T}\times\mathscr{L}$ is the Poincaré group, $\mathscr{T}$ the Abelian group of translations, and $\mathscr{L}=Spin(1,3)$ the Lorentz group. Form the group quotient

$$
\mathscr{P}\times SU(3)\times SU(2)\times U(1)/\mathscr{L}\times SU(3)\times SU(2)\times U(1)
$$

This gives an $\mathscr{L}\times SU(3)\times SU(2)\times U(1)$ fiber bundle over a 4-dim base manifold. We modify this by generalizing the connection, requiring horizontality of the resulting curvatures/field strengths.

Notice that we have chosen the spin representation of the Lorentz generators. This means that when we write an action with this symmetry, we can include spinor fields.

4.1 Generators for the Lie algebra

To carry this out explicitly, we require the Lie algebra of $\mathscr{P}\times SU(3)\times SU(2)\times U(1)$. For $U_Y(1)$ there is only one generator, Y, called hypercharge. We know that $SU(2)$ is generated by the Pauli matrices, which satisfy

$$
[\tau_i,\tau_j]=2i\varepsilon_{ijk}\tau_k
$$

For $SU(3)$, the usual basis is the set of Gell-Mann matrices

$$
\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}
$$

Exercise: Check that these satisfy

$$tr(\lambda_a \lambda_b) = 2\delta_{ab}$$

We will write the commutation relations for the $SU(3)$ generators as

$$[\lambda_a, \lambda_b] = 2i f_{ab}{}^{c} \lambda_c$$

Exercise: Find the commutation relations (i.e., find $f_{ab}{}^{c}$) for the Gell-Mann matrices.

Exercise: Antisymmetry of $f_{ab}{}^{c}$ on the first two indices is automatic, but in fact $f_{abc} = \delta_{cd} f_{ab}{}^{c}$ is totally antisymmetric. Prove this.

Finally, for the Poincaré group we have seen before that we may write

$$\begin{aligned}
[M^a_b, M^c_d] &= -\frac{1}{2}\left(M^a{}_d \eta_{bc} - M^a{}_c \eta_{bd} - M_{bd}\delta^a_c + M_{bc}\delta^a_d\right) \\
[M^a_b, P_c] &= -\frac{1}{2}\left(\delta^a_c P_b - \eta_{bc}\eta^{ad} P_d\right) \\
[P_a, P_b] &= 0
\end{aligned}$$

Since the groups $\mathscr{P}, SU(3), SU(2), U(1)$ are in a direct product, the commutators between the generators from any pair of these different subgroups commute.

4.2 Maurer-Cartan structure equations

From the Lie algebra, we may construct the Maurer-Cartan equations. For each generator we introduce dual 1-forms,

$$\begin{aligned}
\langle Y, \mathbf{B} \rangle &= 1 \\
\left\langle \tau_i, \mathbf{B}^j \right\rangle &= \delta^j_i \\
\left\langle \lambda_a, \mathbf{g}^b \right\rangle &= \delta^b_a \\
\langle M^a_b, \omega^c_d \rangle &= \delta^a_d \delta^c_b - \eta^{ac}\eta_{bd} \\
\left\langle P_a, \mathbf{e}^b \right\rangle &= \delta^b_a
\end{aligned}$$

The Maurer-Cartan equations are immediate,

$$d\omega^A = -\frac{1}{2} c_{BC}{}^{A} \omega^B \wedge \omega^C$$

Substituting the different generators:

$$\begin{aligned}
\mathbf{d}\omega^\alpha{}_\beta &= \omega^\mu{}_\beta \wedge \omega^\alpha{}_\mu \\
\mathbf{d}\mathbf{e}^\alpha &= \mathbf{e}^\beta \wedge \omega^\alpha{}_\beta \\
\mathbf{d}\mathbf{g}^a &= -\frac{1}{2} f_{bc}{}^{a} \mathbf{g}^b \wedge \mathbf{g}^c \\
\mathbf{d}\mathbf{B}^i &= -\frac{1}{2} \varepsilon_{jk}{}^{i} \mathbf{B}^j \wedge \mathbf{B}^k \\
\mathbf{d}\mathbf{B} &= 0
\end{aligned}$$

4.3 Modifying the connections

Now, we generalize the connection 1-forms. This means that the Maurer-Cartan equations will pick up extra, tensorial 2-form terms. These are the field strengths; we require them to be horizontal forms, that is, they expand in the forms $\mathbf{e}^\alpha$ spanning the base manifold only. We will not generalize the Lorentz or translational generators, because we do not yet know how to quantize gravity, and it is difficult to quantize the remaining fields if the background spacetime is curved. Requiring the spacetime curvature and the torsion to vanish means that the solder form and spin connection are still described by the Maurer-Cartan equations,

$$\begin{aligned}
\mathbf{d}\omega^{\alpha}_{\ \beta} &= \omega^{\mu}_{\ \beta}\wedge\omega^{\alpha}_{\ \mu} \\
\mathbf{d}\mathbf{e}^{\alpha} &= \mathbf{e}^{\beta}\wedge\omega^{\alpha}_{\ \beta}
\end{aligned}$$

The first of these shows that the spin connection is pure gauge,

$$\omega^{\alpha}_{\ \beta} = -\mathbf{d}\Lambda^{\alpha}_{\ \mu}\bar{\Lambda}^{\mu}_{\ \beta}$$

where $\Lambda^{\alpha}_{\ \beta}$ is a local Lorentz transformation, and $\bar{\Lambda}^{\alpha}_{\ \beta}$ its inverse. These transformations provide coordinates on the Lorentz part of the fiber bundle, and we may choose them constant. Then the spin connection vanishes and the equation for the solder form reduces to

$$\mathbf{d}\mathbf{e}^{\alpha} = 0$$

The solder form is thus exact. We cannot choose these forms to be zero since they must span the base manifold, but their being exact allows us to write them as differentials of coordinate functions,

$$\mathbf{e}^{\alpha} = \mathbf{d}x^{\alpha}$$

The x^{α} may be taken as our usual Cartesian coordinates on Minkowski space. Since these span the base manifold, all of the (horizontal) curvatures/field strengths must be expanded in terms of them only.

The remaining Maurer-Cartan equations now generalize to

$$\begin{aligned}
\mathbf{d}\mathbf{g}^{a} &= -\frac{1}{2}f_{bc}^{\ \ \ a}\mathbf{g}^{b}\wedge\mathbf{g}^{c}+\mathbf{G}^{a} \\
\mathbf{d}\mathbf{B}^{i} &= -\frac{1}{2}\varepsilon_{jk}^{\ \ \ i}\mathbf{B}^{j}\wedge\mathbf{B}^{k}+\mathbf{F}^{i} \\
\mathbf{d}\mathbf{B} &= \mathbf{H}
\end{aligned}$$

These equations give the usual expressions for the field strengths. For the gluon fields we have

$$\begin{aligned}
\mathbf{G}^{a} &= \mathbf{d}\mathbf{g}^{a}+\frac{1}{2}f_{bc}^{\ \ \ a}\mathbf{g}^{b}\wedge\mathbf{g}^{c} \\
\frac{1}{2}G^{a}_{\alpha\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta} &= \partial_{\alpha}g^{a}_{\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta}+\frac{1}{2}f_{bc}^{\ \ \ a}g^{b}_{\alpha}g^{c}_{\beta}\mathbf{d}x^{\alpha}\wedge\mathbf{d}x^{\beta}
\end{aligned}$$

where $G_{\alpha\beta} = -G_{\beta\alpha}$. Antisymmetrizing the right side and dropping the basis forms,

$$G^{a}_{\alpha\beta} = \partial_{\alpha}g^{a}_{\beta}-\partial_{\beta}g^{a}_{\alpha}+\frac{1}{2}f_{bc}^{\ \ \ a}g^{b}_{\alpha}g^{c}_{\beta}$$

Notice that for an Abelian group, $f_{bc}^{\ \ \ a} = 0$, and this reduces to the same form as the Maxwell field tensor. The extra quadratic terms give the weak and strong interactions considerable richness absent from electromagnetism. Notice also that there is one field for each generator of $SU(3)$.

Exercise: Write the field strengths for the electroweak interaction.

5 The cast of characters

Next, we would like to write an action functional for the standard model. The action we write will require some modification before it can successfully describe the experimental results of particle physics. The first step is to identify possible tensors from which to build the model.

We have the curvatures, $F_{\alpha\beta}, F^{i}_{\ \alpha\beta}$ and $G^{a}_{\ \alpha\beta}$, the solder form, $\mathbf{e}^{\alpha}$, the metric $\eta_{\alpha\beta}$ and the Levi-Civita tensor, $e_{\alpha\beta\mu\nu}$. We also have representations of the groups $\mathscr{L}, SU(3), SU(2), U(1)$. By representations, we mean any tensors of the vector spaces that these groups act on. We have already chosen these.

We have the spinor representation of the Lorentz group, so $\mathscr{L}$ acts on scalars (ϕ), Dirac spinors (ψ^A), bispinors (for example, $X^\alpha = \bar{\psi}\gamma^\alpha\psi, T^{\alpha\beta} = \bar{\psi}\sigma^{\alpha\beta}\psi$) or higher rank tensors such as $S^{ABC} = \psi^A\chi^B\xi^C$.

For $SU(3)$, we have chosen a 3-dimensional representation, and the group can therefore act on the corresponding tensors, $\phi, v^a, T^{ab}, S^{abc}$ and so on, where indices $c_i = (red, blue, green)$ range over colors $c_i = r, b, g$.

The group $SU(2)$ is in a 2-dimensional representation, so we will have tensors built from doublets, $\phi, \xi^{d_1} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, S^{d_1 d_2}$, and higher rank tensors, where $d_i = 1, 2$ for each relevant i.

Finally, $U(1)$ recognizes only the hypercharge, Y, of a field. A group element g acts on a field, ψ, of hypercharge Y as $\psi \rightarrow (g)^Y \psi$.

Since each of these four groups requires two types of index (one ranging over the group generators and one over the components of the matrix representation), we quickly run out of alphabets. Therefore, wherever possible, we will suppress the matrix/vector indices.

The notation becomes tricky because one object may have any or all of these labels. For example, the up and down quark together form an $SU(2)$ doublet,

$$\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

Each component of this doublet is a Lorentz spinor of definite color,

$$u = \psi^A_{c_1}; d = \chi^B_{c_2}$$

and has an associated hypercharge $Y = \frac{1}{3}$.

6 A trial action

The most straightforward action we can write is to include Yang-Mills type terms for the field strengths, and multiplets for the spinors and leptons. If we want to be fancy, we can write the Yang-Mills action as

$$S = \frac{1}{8}\int tr(\mathbf{G}^*\mathbf{G})$$

where

$$\mathbf{G} = G^a_{\alpha\beta}[\lambda_a]$$

and the trace is taken over the product of the λs. Substituting gives

$$S_{gluons} = \frac{1}{4}\int G^a_{\alpha\beta}G^{a\alpha\beta}d^4x$$

as the action for the color gauge bosons, called gluons. Here we sum on the a index, even though both are written up. We can get away with this because the metric is just δ_{ab}, so there is no difference between raised and lowered a indices.

The quarks are spin-$\frac{1}{2}$ Dirac particles, so their action should be

$$S_{quarks} = \sum_c\sum_q\int \bar{\psi}_q(i\not{D} - m)\psi_q$$

where the sum on c runs over colors and the sum on q runs over the set of quarks, $(u, d, c, s, t, b, \ldots)$ and the derivative operator, D, is covariant with respect to the relevant symmetries.

Later, we will discuss an approximate unitary symmetry – flavor symmetry – among the different quarks. This symmetry allows the interchange of one quark for another, producing a distinct particle state. It has been very successful at making sense of the many possible states of bound quarks. But for now, we restrict attention to color symmetry.

For the weak interaction, we start with the free action for the electroweak gauge bosons,

$$S_{\gamma,W^\pm,Z^0} = \frac{1}{4}\int\left(F^i_{\alpha\beta}F^{i\alpha\beta} + H_{\alpha\beta}H^{\alpha\beta}\right)d^4x$$

To this, we add the action for the leptons,

$$S_{leptons} = \sum_{leptons} \int \bar{\psi} \left(i\not{D} - m \right) \psi$$

where the $l \in \left(e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau, \ldots\right)$ and the derivative is again covariant.

There are several things wrong with this simple picture:

1. Three of the electroweak gauge bosons are massive. This problem is particularly acute, since adding an ordinary mass term would break the $SU(2) \times U(1)$ gauge symmetry. This leads to the introduction of the Higgs particle, a complex pair of Lorentz scalar fields transforming as an electroweak doublet.

2. The weak interaction violates parity, but the interaction term in $S_{leptons}$ does not. The electromagnetic interaction preserves parity.

3. The fermions all must have the same mass, or the unitary symmetries between fermions are lost.

4. The (u, d, c, s, t, b) quarks in the quark action are not the states that couple to the weak interaction. Instead, certain linear combinations of these spinor fields couple to the weak gauge bosons.

Since these difficulties involve the weak interaction only, we treat the strong interaction first.

7 Strong interactions

The action for the strong interaction is the sum of the parts for the gauge bosons and the quarks:

$$S_{strong} = \frac{1}{4} \sum_{a=1}^{8} \int G^a_{\alpha\beta} G^{a\alpha\beta} d^4x + \sum_{c=r,g,b} \sum_q \int \bar{\psi}_q \left(i\not{D} - m \right) \psi_q$$

where for each $q \in \{u, d, c, s, t, b\}$, ψ_q is an $SU(3)$ triplet of spinor fields. The covariant derivative is

$$\mathbf{D}\psi_q = \mathbf{d}\psi_q - g_s \mathbf{g}^a \lambda_a \psi_q + electroweak$$

where the strong coupling constant, g_s, characterizes the strength of the strong interaction and we specify the electroweak part of the connection in the next section. The gluon field strength is

$$G^a_{\alpha\beta} = \partial_\alpha g^{\ a}_{\beta} - \partial_\beta g^{\ a}_{\alpha} + g_s f_{bc}^{\ \ \ a} g^{\ b}_{\alpha} g^{\ c}_{\beta}$$

Now consider the various generators of $SU(3)$. They act on color triplets of the form

$$\psi_q = \begin{pmatrix} \psi_{q,r} \\ \psi_{q,g} \\ \psi_{q,b} \end{pmatrix}$$

For example, consider the interaction term in the Lagrangian containing the generator $g^1 \lambda_1$,

$$\begin{aligned}
\bar{\psi}_q \left(i\not{g}^1 \lambda_1 \right) \psi_q &= i g^1_\alpha \bar{\psi}_q \gamma^\alpha \left(\lambda_1 \psi_q \right) \\
&= i g^1_\alpha \bar{\psi}_q \gamma^\alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{q,r} \\ \psi_{q,g} \\ \psi_{q,b} \end{pmatrix} \\
&= i g^1_\alpha \begin{pmatrix} \bar{\psi}_{q,r} & \bar{\psi}_{q,g} & \bar{\psi}_{q,b} \end{pmatrix} \gamma^\alpha \begin{pmatrix} \psi_{q,g} \\ \psi_{q,r} \\ 0 \end{pmatrix} \\
&= i g^1_\alpha \left(\bar{\psi}_{q,r} \gamma^\alpha \psi_{q,g} + \bar{\psi}_{q,g} \gamma^\alpha \psi_{q,r} \right)
\end{aligned}$$

When the spinors are expanded in the usual way, the term

$$ig^1_\alpha \bar{\psi}_{q,r}\gamma^\alpha \psi_{q,g}$$

can create an antigreen quark or annihilate a green quark, while creating a red quark or annihilating an antired quark. The gluon described by the vector field g^1_α therefore carries off one unit of green and one unit of antired, and may be called a green/antired gluon. The second term, $ig^1_\alpha \bar{\psi}_{q,g}\gamma^\alpha \psi_{q,r}$, has the opposite effects and therefore describes a red/antigreen gluon – the antiparticle of the green/antired.

There are three distinct types of interaction determined by the gluon term in the action. Expanding the first term of the Lagrangian gives

$$\begin{aligned}
\frac{1}{4}\sum_{a=1}^{8} G^a_{\alpha\beta}G^{a\alpha\beta} &= \frac{1}{4}\sum_{a=1}^{8}\eta^{\alpha\mu}\eta^{\beta\nu}\left(\partial_\alpha g^a_\beta - \partial_\beta g^a_\alpha + gf_{bc}{}^{a}g^b_\alpha g^c_\beta\right)\left(\partial_\mu g^a_\nu - \partial_\nu g^a_\mu + gf_{bc}{}^{a}g^b_\mu g^c_\nu\right) \\
&= \frac{1}{4}\sum_{a=1}^{8}\eta^{\alpha\mu}\eta^{\beta\nu}\left(\partial_\alpha g^a_\beta - \partial_\beta g^a_\alpha\right)\left(\partial_\mu g^a_\nu - \partial_\nu g^a_\mu\right) \\
&\quad + g\sum_{a=1}^{8}\left(\eta^{\alpha\mu}\eta^{\beta\nu}f_{bc}{}^{a}\right)g^b_\alpha g^c_\beta \partial_\mu g^a_\nu \\
&\quad + \frac{g^2}{4}\sum_{a=1}^{8}\left(\eta^{\alpha\mu}\eta^{\beta\nu}f_{bc}{}^{a}f_{de}{}^{a}\right)g^b_\alpha g^c_\beta g^d_\mu g^e_\nu
\end{aligned}$$

The first term on the right is just like the free electromagnetic field, and describes an uncoupled gluon. Its quantization leads to the gluon propagator.

The remaining two terms are interactions. The first involves three gluons and contains a derivative. When expressed in momentum space, the derivative makes the vertex contribution proportional to the momentum. The total antisymmetry of f_{abc} then means that the vertex contribution to the matrix element, when the gluons have momenta p_β, q_α, r_μ is proportional to

$$gf_{abc}\left(\eta_{\alpha\beta}(p-q)_\mu + \eta_{\mu\alpha}(q-r)_\beta + \eta_{\beta\mu}(r-p)_\alpha\right)$$

The final interaction involves four gluons and two structure constants. Letting the gluons have indices $g^a_\alpha, g^b_\beta, g^c_\mu, g^d_\nu$ leads to a vertex contribution of

$$-ig^2\left[f_{eab}f_{ecd}\left(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}\right) + f_{eac}f_{edb}\left(\eta_{\alpha\nu}\eta_{\mu\beta} - \eta_{\alpha\beta}\eta_{\mu\nu}\right) + f_{ead}f_{ebc}\left(\eta_{\alpha\beta}\eta_{\nu\mu} - \eta_{\alpha\mu}\eta_{\nu\beta}\right)\right]$$

These same vertex contributions occur for any non-Abelian Yang-Mills theory, the only difference being the value of the coupling constant, g.

8 Electroweak Interactions

So far, we have the electroweak action

$$S_{electroweak} = \frac{1}{4}\int F^i_{\alpha\beta}F^{i\alpha\beta}d^4x + \sum_{leptons}\int \bar{\psi}\left(i\not{D} - m\right)\psi$$

together with the coupling in the quark covariant derivative, where the $l \in \left(e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau, \ldots\right)$. The covariant derivative is that for $SU(2)\times U(1)$, so

$$D_\alpha\psi = \partial_\alpha\psi + \frac{i}{2}gB^i_\alpha\tau_i\psi + \frac{i}{2}g'YB_\alpha\psi$$

We need to correct three problems:

1. Three of the electroweak gauge bosons are massive, while the remaining one (the photon) is massless. This problem is particularly acute, since adding an ordinary mass term, $\sum_i\left(m_i^2 B^i_\alpha B^{i\alpha}\right)$ would break the $SU(2)\times U(1)$ gauge symmetry. This leads to the introduction of the Higgs particle, a complex pair of Lorentz scalar fields transforming as an electroweak doublet.

2. The weak interaction violates parity, but the interaction term in $S_{leptons}$ does not.
3. The (u,d,c,s,t,b) quarks in the quark action are not the states that couple to the weak interaction. Instead, certain linear combinations of these spinor fields couple to the weak gauge bosons.

We consider these in turn.

8.1 Massive gauge bosons

8.1.1 Spontaneous symmetry breaking

The problem of massive gauge particles can be solved by introducing additional fields. The basic idea is to introduce a scalar field with a quartic potential of the general form

$$S = \int \partial_\alpha \phi \partial^\alpha \phi + \alpha \phi^2 - \beta \phi^4$$

Notice that S is invariant under the discrete symmetry, $\phi \to -\phi$. However, the potential insures that low energy solutions will break this symmetry, because the field will fall to a minimum of the potential, $V = -\alpha\phi^2 + \beta\phi^4$. The extrema of V occur at

$$\begin{aligned} 0 &= \frac{dV}{d\phi} \\ &= -2\alpha\phi + 4\beta\phi^3 \\ &= \left(-2\alpha + 4\beta\phi^2\right)\phi \end{aligned}$$

We choose $\lambda > 0$ so that the extremum at $\phi = 0$ is a local maximum; minima then occur at

$$\phi_0 = \pm\sqrt{\frac{\alpha}{2\beta}}$$

where the potetial has the value

$$\begin{aligned} V(\phi_0) &= -\alpha\phi_0^2 + \beta\phi_0^4 \\ &= -\frac{\alpha^2}{2\beta} + \frac{\alpha^2}{4\beta} \\ &= -\frac{\alpha^2}{4\beta} \end{aligned}$$

For energies small relative to $\frac{\alpha^2}{4\beta}$, we may expand the field about either of these minima. Thus, if we write

$$\phi = \phi_0 + \eta$$

then

$$\begin{aligned} S &= \int \partial_\alpha \eta \partial^\alpha \eta + m^2 (\phi_0 + \eta)^2 - \lambda (\phi_0 + \eta)^4 \\ &= \int \partial_\alpha \eta \partial^\alpha \eta + \left(\alpha\phi_0^2 - \beta\phi_0^4\right) + \left(2\alpha\phi_0 - 4\beta\phi_0^3\right)\eta + \left(\alpha - 6\beta\phi_0^2\right)\eta^2 - 4\beta\phi_0\eta^3 - \eta^4 \\ &= \int \partial_\alpha \eta \partial^\alpha \eta + \frac{\alpha^2}{4\beta} - 2\alpha\eta^2 \mp \sqrt{8\alpha\beta}\eta^3 - \eta^4 \end{aligned}$$

and, dropping the irrelevant constant, the effective action is that of a Klein-Gordon field with mass 2α, having cubic and quartic self-interactions.

$$S = \int \partial_\alpha \eta \partial^\alpha \eta - 2\alpha\eta^2 \mp \sqrt{8\alpha\beta}\eta^3 - \eta^4$$

This is an example of spontaneous symmetry breaking. In this case, as an action which is invariant under the replacement $\phi \to -\phi$ finds its way to a solution which lacks this symmetry.

The usefulness of the symmetry breaking is that we can use the vacuum expectation value, ϕ_0, of the original field as a mass for another particle. For example, suppose we had a second, massless field, A_α, with a coupling term $\phi A_\alpha A^\alpha$. Then the expansion of ϕ about ϕ_0 would give an effective mass term $\phi_0 A_\alpha A^\alpha$ to field A_α.

8.1.2 The Higgs Mechanism

Our goal now is to write a symmetry breaking term for the electroweak gauge fields in a gauge invariant way, by letting ϕ have an $SU(2)$ symmetry. We can accomplish this by making ϕ into a complex $SU(2)$ doublet,

$$\phi^a = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

and assigning it a hypercharge of -1. We need ϕ^a complex so that $|\phi|^2 = \phi^{\dagger a}\phi^a$ is acted on nontrivially by $SU(2)$. With this, the action ϕ may be conveniently written as

$$S = \int (D_\alpha \phi^a)^\dagger (D^\alpha \phi^a) + \lambda \left(|\phi|^2 - v^2\right)^2$$

the extra constant, λv^4 has no effect. Notice that the potential now has an continuum of minima at $|\phi|^2 = v^2$. Motion along this continuum requires no energy, and therefore corresponds to a massless excitation of the Higgs field called a Goldstone boson.

Expanding the $SU(2)$-covariant derivative and setting the hypercharge to $Y = -1$, gives

$$\begin{aligned}
S &= \int \eta^{\alpha\beta}\left(\partial_\alpha \phi^{a\dagger} - \frac{i}{2} g B^i_\alpha \left(\phi^\dagger \tau_i\right)^a - \frac{i}{2} g' B_\alpha \phi^{a\dagger} Y\right)\left(\partial_\beta \phi^a + \frac{i}{2} g B^j_\beta \left(\tau_j \phi\right)^a + \frac{i}{2} g' B_\beta Y \phi^a\right) + \lambda\left(|\phi|^2 - v^2\right)^2 \\
&= \int \eta^{\alpha\beta}\partial_\alpha \phi^{a\dagger}\partial_\beta \phi^a - \frac{i}{2} g \eta^{\alpha\beta} B^i_\alpha \left(\phi^\dagger \tau_i \partial_\beta \phi\right) + \frac{i}{2} g' \eta^{\alpha\beta} B_\alpha \left(\phi^\dagger \partial_\beta \phi\right) \\
&\quad + \frac{i}{2} g \eta^{\alpha\beta} B^j_\beta \left(\partial_\alpha \phi^{a\dagger} \tau_j \phi\right) + \frac{1}{4} g^2 \eta^{\alpha\beta} B^i_\alpha B^j_\beta \left(\phi^\dagger \tau_i \tau_j \phi\right) - \frac{1}{4} g g' \eta^{\alpha\beta} B_\alpha B^j_\beta \left(\phi^\dagger \tau_j \phi\right) \\
&\quad - \frac{i}{2} g' \eta^{\alpha\beta} B_\beta \left(\partial_\alpha \phi^\dagger \phi\right) - \frac{1}{4} g g' \eta^{\alpha\beta} B^i_\alpha B_\beta \left(\phi^\dagger \tau_i \phi\right) + \frac{1}{4} g'^2 \eta^{\alpha\beta} B_\alpha B_\beta \left(\phi^\dagger \phi\right) \\
&\quad + \lambda\left(|\phi|^2 - v^2\right)^2
\end{aligned}$$

It is the terms quadratic in the gauge fields B^i_α, B_β,

$$\begin{aligned}
\mathscr{L}_m &= \frac{1}{4} g^2 \eta^{\alpha\beta} B^i_\alpha B^j_\beta \left(\phi^\dagger \tau_i \tau_j \phi\right) - \frac{1}{4} g g' \eta^{\alpha\beta} B_\alpha B^j_\beta \left(\phi^\dagger \tau_j \phi\right) \\
&\quad - \frac{1}{4} g g' \eta^{\alpha\beta} B^i_\alpha B_\beta \left(\phi^\dagger \tau_i \phi\right) + \frac{1}{4} g'^2 \eta^{\alpha\beta} B_\alpha B_\beta \left(\phi^\dagger \phi\right)
\end{aligned}$$

that we must study to see what combinations of fields acquire mass.

8.1.3 Choosing a gauge

To simplify the problem, we may choose the local $SU(2)$ gauge any way we like. The greatest simplification occurs if we use it to restrict the form of ϕ^a.

Under an $SU(2)$ transformation, ϕ^a changes according to

$$\begin{aligned}
\phi^a &\rightarrow \left[e^{\frac{i\varphi}{2}\mathbf{n}\cdot\sigma}\right]^a{}_b \phi^b \\
&= \phi^a \cos\frac{\varphi}{2} + i\left[\mathbf{n}\cdot\sigma\right]^a{}_b \phi^b \sin\frac{\varphi}{2} \\
&= \begin{pmatrix} \phi^1\left(\cos\frac{\varphi}{2} + i n_z \sin\frac{\varphi}{2}\right) + \phi^2\left(i n_x + n_y\right)\sin\frac{\varphi}{2} \\ \phi^2\left(\cos\frac{\varphi}{2} - i n_z \sin\frac{\varphi}{2}\right) + \phi^1\left(i n_x - n_y\right)\sin\frac{\varphi}{2} \end{pmatrix}
\end{aligned}$$

Now let $\alpha = (n_x + i n_y)\sin\frac{\varphi}{2}$ and $\beta = \cos\frac{\varphi}{2} - i n_z \sin\frac{\varphi}{2}$, where $|\alpha|^2, |\beta|^2 \leq 1$, and demand

$$\begin{aligned}
0 &= \beta\phi^2 + i\alpha\phi^1 \\
\alpha &= \frac{i\phi^2}{\phi^1}\beta
\end{aligned}$$

so we can make the second component vanish. With this ratio of α we may still choose any β, so consider the first component

$$\begin{aligned}\phi^1 &\rightarrow \bar{\beta}\phi^1 + i\bar{\alpha}\phi^2 \\ &= \bar{\beta}\left(\phi^2 + i\phi^1\frac{\bar{\alpha}}{\bar{\beta}}\right) \\ &= \bar{\beta}\left(\phi^2 + \phi^1\frac{\bar{\phi}^2}{\bar{\phi}^1}\right) \\ &= \frac{\bar{\beta}}{\bar{\phi}^1}\left(\phi^2\bar{\phi}^1 + \phi^1\bar{\phi}^2\right)\end{aligned}$$

The magnitude of β must be less than one, but we can choose the phase of β so that ϕ^2 is real. Now ϕ^a has the form

$$\phi^a = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

with f real, and since the $SU(2)$ symmetry is local, we can achieve this form at each point. We expand the remaining field about the minimum, $|\phi|^2 = v^2$,

$$f = v + \eta$$

8.1.4 Masses for the gauge bosons

We now work out the quadratic factors at the minimum, $\phi_0^a = \begin{pmatrix} v \\ 0 \end{pmatrix}$:

$$\begin{aligned}\phi_0^\dagger \tau_i \tau_j \phi_0 &= v^2\delta_{ij} + i\varepsilon_{ijk}\phi_0^\dagger\tau_k\phi_0 \\ &= v^2\left(\delta_{ij} + i\varepsilon_{ij3}\right) \\ \phi_0^\dagger Y \tau_j \phi_0 &= \delta_{j3}v^2 \\ Y^2\phi_0^\dagger\phi_0 &= v^2\end{aligned}$$

Now substitute into the quadratic part of the Lagrange density for the masses,

$$\begin{aligned}\mathscr{L}_m &= \frac{1}{4}g^2v^2B^i_\alpha B^{i\alpha} - \frac{1}{4}gg'v^2B^\alpha B^3_\alpha \\ &\quad -\frac{1}{4}gg'v^2B^3_\alpha B^\alpha + \frac{1}{4}g'^2v^2B_\alpha B^\alpha\end{aligned}$$

Define the ratio of the $SU(2)$ and $U_Y(1)$ coupling constants to be

$$\frac{g'}{g} = \tan\theta_W$$

Then

$$\begin{aligned}\mathscr{L}_m &= \frac{1}{4}g^2v^2\left(B^1_\alpha B^{1\alpha} + B^2_\alpha B^{2\alpha}\right) + \frac{1}{4}g^2v^2\left(B^3_\alpha B^{3\alpha} - B^\alpha B^3_\alpha\tan\theta_W - B^3_\alpha B^\alpha\tan\theta_W + B_\alpha B^\alpha\tan^2\theta_W\right) \\ &= \frac{1}{4}g^2v^2\left(B^1_\alpha B^{1\alpha} + B^2_\alpha B^{2\alpha}\right) \\ &\quad + \frac{g^2v^2}{4\cos^2\theta_W}\left(B^3_\alpha B^{3\alpha}\cos^2\theta_W - B^\alpha B^3_\alpha\sin\theta_W\cos\theta_W - B^3_\alpha B^\alpha\sin\theta_W\cos\theta_W + B_\alpha B^\alpha\sin^2\theta_W\right) \\ &= \frac{1}{4}g^2v^2\left(B^1_\alpha B^{1\alpha} + B^2_\alpha B^{2\alpha}\right) + \frac{g^2v^2}{4\cos^2\theta_W}\left(B^3_\alpha\cos\theta_W - B_\alpha\sin\theta_W\right)^2\end{aligned}$$

Notice that

$$B^i_\alpha\tau_i = \begin{pmatrix} B^3_\alpha & B^1_\alpha - iB^2_\alpha \\ B^1_\alpha + iB^2_\alpha & -B^3_\alpha \end{pmatrix}$$

so that we may combine B^1 and B^2 into the complex conjugate states

$$\begin{aligned} W_\alpha^+ &= B_\alpha^1 + iB_\alpha^2 \\ W_\alpha^- &= B_\alpha^1 - iB_\alpha^2 \end{aligned}$$

Then W^+ is a charged particle with antiparticle W^-. Since

$$W_\alpha^+ W^{-\alpha} = B_\alpha^1 B^{1\alpha} + B_\alpha^2 B^{2\alpha}$$

the W$^\pm$ mass is $m_W = \frac{gv}{2}$.

For the remaining two gauge fields, B_α^3 and B_α, we identify the orthogonal combinations

$$\begin{aligned} Z_\alpha^0 &= B_\alpha^3 \cos\theta_W - B_\alpha \sin\theta_W \\ A_\alpha &= B_\alpha^3 \sin\theta_W + B_\alpha \cos\theta_W \end{aligned}$$

The Z^0 is the only combination appearing in $\mathscr{L}_m$, and it has mass $m_Z = \frac{gv}{2\cos\theta_W}$. The combination of B_α^3 and B_α orthogonal to Z_α^0 is absent from $\mathscr{L}_m$ and therefore describes a particle which remains massless. We identify this combination with the photon.

The Weinberg angle is now *defined* as the ratio between the W and Z^0 masses,

$$\begin{aligned} \cos\theta_W &= \frac{m_W}{m_{Z_0}} \\ m_W &= 80.398 \pm .025 \\ m_{Z^0} &= 91.1876 \pm .0021 \\ \theta_W &\approx 30^\circ \end{aligned}$$

8.1.5 Masses for fermions

We also require the Higgs mechanism to give different masses to fermions. The action we have written so far describes fermions with equal masses,

$$\begin{aligned} S_{leptons} &\sim \sum_{leptons} \int \bar{\psi}\left(i\not{D} - m\right)\psi \\ S_{quarks} &\sim \sum_{c=r,g,b} \sum_{quarks} \int \bar{\psi}\left(i\not{D} - m\right)\psi \end{aligned}$$

As noted above, if we made the masses different in these expressions, then the mass terms would not allow $SU(2)$ or $SU(3)$ rotations among the corresponding fermions. However, we know these masses to be different.

We can use the Higgs mechanism to change the different quark and lepton masses when the Higgs settles near its vacuum expectation value. However, the required coupling hinges on having the correct fields, including the appropriate parity combinations. Once we discuss parity in the next section, we will return to the question of fermion masses.

8.2 Parity

We know that the weak interaction violates parity, but the electromagnetic interaction preserves parity. This means that we need to introduce an asymmetry between left and right spinor fields, given by using the projection operators, $L = \frac{1}{2}(1-\gamma_5), R = \frac{1}{2}(1+\gamma_5)$.

To correct the parity problem, first consider the parity of the various Lorentz tensors built as bispinors:

$$\begin{array}{rl} \bar{\psi}\psi & scalar \\ \bar{\psi}\gamma^\alpha\psi & vector \\ \bar{\psi}\sigma^{\alpha\beta}\psi & tensor \\ \bar{\psi}\gamma^\alpha\gamma_5\psi & pseudovector \\ \bar{\psi}\gamma_5\psi & pseudoscalar \end{array}$$

Each of these has definite parity, so if we want to violate parity, we need a combination of two or more. The vector already occurs in the action, coupled to the gauge bosons via the connection term of the covariant derivative

$$S_{electroweak} = \frac{1}{4}\int F^i_{\alpha\beta}F^{i\alpha\beta}d^4x + \sum_{leptons}\int \bar{\psi}\left(i\not{D} - m\right)\psi$$

The coupling term comes from the derivative of ψ. If ψ is an $SU(2)$ doublet then

$$i\bar{\psi}\not{D}\psi = i\bar{\psi}\left(\gamma^\beta\partial_\beta\psi + \frac{i}{2}g\gamma^\beta B^j_\beta\tau_j\psi + \frac{i}{2}g'\gamma^\beta B_\beta Y\psi\right)$$

From Noether's theorem, we know that the currents are

$$\begin{aligned} J^{i\beta} &= -\frac{1}{2}g\bar{\psi}\gamma^\beta\tau^i\psi \\ J^\beta_Y &= -\frac{1}{2}g'\bar{\psi}\gamma^\beta Y\psi \end{aligned}$$

so we see that the couplings between the gauge fields and the currents are of the same form as in Maxwell theory,

$$B^i_\beta J^{i\beta}, B_\beta J^\beta_Y$$

Notice that J^i_β is a vector current. The simplest way to violate parity is to take a linear combination of the vector and pseudovector (or axial vector) currents.

$$J^\alpha_i = \alpha g\bar{\psi}\gamma^\alpha\tau_i\psi + \beta g\bar{\psi}\gamma^\alpha\gamma_5\tau_i\psi$$

Of course, any combination of the Lorentz bispinors is allowed in principle, but this one turns out to be right. In fact, the best agreement with experiment occurs if we choose the constants α, β so that the violation is maximal, $\alpha = -\beta = \frac{1}{2}$. This gives

$$\begin{aligned} J^\alpha_i &= g\bar{\psi}\gamma^\alpha\frac{1}{2}\left(1-\gamma_5\right)\tau_i\psi \\ &= g\bar{\psi}_L\gamma^\alpha\tau_i\psi_L \end{aligned}$$

where ψ_L is a left-handed doublet, for example, $\psi_L = \frac{1}{2}\left(1-\gamma_5\right)\psi = \begin{pmatrix} e_L \\ \nu_L \end{pmatrix}$. Because this involves the vector (V) and axial vector (A) this is often called the $V - A$ coupling. The left-handed combination $\frac{1}{2}\left(1-\gamma_5\right)$ is a projection operator since it is idempotent:

$$\begin{aligned} \frac{1}{2}\left(1-\gamma_5\right)\frac{1}{2}\left(1-\gamma_5\right) &= \frac{1}{4}\left(1-\gamma_5-\gamma_5+\gamma_5\gamma_5\right) \\ &= \frac{1}{2}\left(1-\gamma_5\right) \end{aligned}$$

The complementary, right-handed projection operator is $\frac{1}{2}\left(1+\gamma_5\right)$.

In order for the electromagnetic part of the interaction to preserve parity, we must also include some right handed fields. These we take to be *scalars under SU(2)*, for example

$$e_R = \frac{1}{2}\left(1+\gamma_5\right)e$$

We introduce only the right-handed electron, muon and tau, but not the right-handed neutrinos because the neutrinos only participate in the weak, parity violating part of the interaction.

Having the electron, muon, and tau represented by a right-handed scalar and a left-handed doublet allows us to couple these fields to the Higgs doublet with *Yukawa couplings*. Let

$$\begin{aligned} \psi^L_e &= \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} \\ \psi^R_e &= e_R \end{aligned}$$

where e_R, e_L and ν_L are the right- and left-handed electron and the left-handed neutrino spinors. We make this a Lorentz scalar and $SU(2)$ doublet by contracting with $\bar{\psi}_e^R$,

$$\bar{\psi}_e^R \psi_e^L = \begin{pmatrix} \bar{e}_R e_L \\ \bar{e}_R \nu_L \end{pmatrix}$$

Then the Yukawa coupling for the electron is the product of this scalar doublet with the Higgs scalar doublet:

$$G_e \left(\phi^\dagger \bar{e}_R \psi_L + \bar{\psi} e_R \phi\right) = G_e \left(\phi_1^*, \phi_2^*\right) \begin{pmatrix} \bar{e}_R e_L \\ \bar{e}_R \nu_L \end{pmatrix} + G_e \left(\bar{e}_L e_R, \bar{\nu}_L e_R\right) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

When the Higgs particle equals its vacuum expectation value,

$$\phi = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

Then this term becomes

$$G_e \left(\phi^\dagger \bar{e}_R \psi_L + \bar{\psi} e_R \phi\right) = G_e v \left(\bar{e}_R e_L + \bar{e}_L e_R\right)$$

This is exactly the form of a fermion mass term, with $m_e = G_e v$ for the electron field. We introduce similar Yukawa terms for the muon and tau, but with coupling constants G_μ and G_τ, giving a total Yukawa interaction of

$$\sum_{l=e,\mu,\tau} G_l \left(\phi^\dagger \bar{\psi}_l^R \psi_l^L + \bar{\psi}_l^L \psi_l^R \phi\right)$$

8.3 The electroweak Lagrangian for leptons

We may now write the Lagrange density for the gauge particles and leptons of the full electroweak interaction. In condensed form, we see the essential features:

$$\begin{aligned} \mathscr{L}_{EW,leptons} &= \frac{1}{4} F^i_{\alpha\beta} F^{i\alpha\beta} + \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta} \\ &\quad + \sum_{l=e,\mu,\tau} i \bar{\psi}_l^L \not{D}^{1,2} \psi_l^L + \sum_{l=e,\mu,\tau} i \bar{\psi}_l^R \not{D}^1 \psi_l^R \\ &\quad + \left|D_\beta \phi\right|^2 + \lambda \left(|\phi|^2 - v^2\right)^2 + \sum_{l=e,\mu,\tau} G_l \left(\phi^\dagger \bar{\psi}_l^R \psi_l^L + \bar{\psi}_l^L \psi_l^R \phi\right) \end{aligned}$$

The first line contains the kinetic and self-interaction terms for the $W^\pm, Z^0$ and photon; the second line gives the kinetic term for the leptons, where $D^{1,2}$ is the $SU(2) \times U(1)$ covariant derivative and D^1 is the $U(1)$ covariant derivative; the final line gives the kinetic term, potential, and Yukawa couplings for the Higgs particle.

The first thing we need to do is to rearrange the gauge couplings for the photon and weak fields,

$$\begin{aligned} A_\alpha &= B^3_\alpha \sin\theta_W + B_\alpha \cos\theta_W \\ W^+_\alpha &= B^1_\alpha + i B^2_\alpha \\ W^-_\alpha &= B^1_\alpha - i B^2_\alpha \\ Z^0_\alpha &= B^3_\alpha \cos\theta_W - B_\alpha \sin\theta_W \end{aligned}$$

to the fermion spinors. This willl allow us to identify the electron, muon and tau, since it is only the these leptons that couple to the photon. Expanding the covariant derivative term, we have

$$
\begin{aligned}
i\bar{\psi}\not{D}^{1,2}\,\psi^L &= i\bar{\psi}^L\gamma^\beta\left(\partial_\beta\psi+\frac{i}{2}gB^j_\beta\tau_j\psi+\frac{i}{2}g'B_\beta Y\psi\right)\\
&= i\bar{\psi}^L\gamma^\beta\left(\partial_\beta\psi+\frac{i}{2}gB^1_\beta\tau_1\psi+\frac{i}{2}gB^2_\beta\tau_2\psi+\frac{i}{2}gB^3_\beta\tau_3\psi+\frac{i}{2}g'B_\beta Y\psi\right)\\
&= i\bar{\psi}^L\gamma^\beta\left(\partial_\beta\psi+\frac{i}{4}g\left(W^+_\beta+W^-_\beta\right)\tau_1\psi+\frac{i}{4i}g\left(W^+_\beta-W^-_\beta\right)\tau_2\psi\right)\\
&\quad +i\bar{\psi}^L\gamma^\beta\left(\frac{ig}{4}\left(A_\beta\sin\theta_W+Z^0_\beta\cos\theta_W\right)\tau_3\psi+\frac{i}{4}g'\left(A_\beta\cos\theta_W-Z^0_\beta\sin\theta_W\right)Y\psi\right)\\
&= i\bar{\psi}^L\gamma^\beta\left(\partial_\beta\psi+\frac{i}{4}g\left(W^+_\beta\left(\tau_1-i\tau_2\right)+W^-_\beta\left(\tau_1+i\tau_2\right)\right)\psi\right)\\
&\quad -\frac{g}{4\cos\theta_W}\bar{\psi}^L\gamma^\beta Z^0_\beta\left(\tau_3\cos^2\theta_W\psi-Y\sin^2\theta_W\right)\psi\\
&\quad -\frac{g\sin\theta_W}{4}\bar{\psi}^L\gamma^\beta A_\beta\left(\tau_3+Y\right)\psi
\end{aligned}
$$

We can write $\tau_3=\frac{1}{2}\sigma_3$ as a quantum number, $I=\pm\frac{1}{2}$, for isospin, times the identity. Write isospin doublets as lepton/neutrino pairs, $\begin{pmatrix} l \\ \nu \end{pmatrix}$, where $I_3=+\frac{1}{2}$ for $l=e,\mu,\tau$ and $I_3=-\frac{1}{2}$ for $\nu=\nu_e,\nu_\mu,\nu_\tau$. We also replace $Y=\frac{Y_W}{2}$, to agree with standard usage. Then we have

$$
\begin{aligned}
i\bar{\psi}\not{D}^{1,2}\,\psi^L &= i\bar{\psi}^L\gamma^\beta\left(\partial_\beta\psi+\frac{i}{4}g\left(W^+_\beta\left(\tau_1-i\tau_2\right)+W^-_\beta\left(\tau_1+i\tau_2\right)\right)\psi\right)\\
&\quad -\frac{g}{4\cos\theta_W}\bar{\psi}^L\gamma^\beta Z^0_\beta\left(I_3\cos^2\theta_W\psi-Y\sin^2\theta_W\right)\psi\\
&\quad -\frac{g\sin\theta_W}{4}\bar{\psi}^L\gamma^\beta A_\beta\left(I_3+\frac{Y_W}{2}\right)\psi
\end{aligned}
$$

and we may identify the unit electric charge $e=g\sin\theta_W$,

$$q=e\left(I+\frac{Y_W}{2}\right)$$

and the electromagnetic current as

$$
\begin{aligned}
J_{EM} &= -\frac{g\sin\theta_W}{4}\bar{\psi}\gamma^\beta\left(I_3+\frac{Y_W}{2}\right)\psi_L\\
&= -\frac{q}{4}\bar{\psi}\gamma^\beta\psi_L
\end{aligned}
$$

where the extra factor of $-\frac{1}{4}$ is due to a bad choice of initial normalization.

The leptons are assigned a weak hypercharge quantum number of $Y_W=1$, so that the e,μ,τ have an electric charge of e while the neutrino fields have $q=0$.

Also, notice that

$$
\begin{aligned}
\tau_1-i\tau_2 &= \frac{1}{2}\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix}+\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\right)\\
&= \begin{pmatrix} & 0\\ 1 & \end{pmatrix}
\end{aligned}
$$

so the current coupling to the W^- is

$$
\begin{aligned}
J^\beta_+ &= -\frac{g}{4}\bar{\psi}\gamma^\beta\left(\tau_1-i\tau_2\right)\psi_L\\
&= -\frac{g}{4}\bar{\nu}\gamma^\beta l_L
\end{aligned}
$$

while

$$\begin{aligned} J_-^\beta &= -\frac{g}{4}\bar{\psi}\gamma^\beta(\tau_1+i\tau_2)\psi_L \\ &= -\frac{g}{4}\bar{l}\gamma^\beta\nu_L \end{aligned}$$

couples to the W^+.

The current coupling to the Z^0 is given by

$$\begin{aligned} J_Z^\beta &= -\frac{g}{4\cos\theta_W}\bar{\psi}^L\gamma^\beta\left(I_3\cos^2\theta_W - Y\sin^2\theta_W\right)\psi \\ &= -\frac{g}{4\cos\theta_W}\bar{\psi}^L\gamma^\beta\left(I_3\left(\cos^2\theta_W+\sin^2\theta_W\right) - I_3\sin^2\theta_W - Y\sin^2\theta_W\right)\psi \\ &= -\frac{g}{4\cos\theta_W}\bar{\psi}^L\gamma^\beta\left(I_3 - (I_3+Y)\sin^2\theta_W\right)\psi \end{aligned}$$

$$\begin{aligned} &= -\bar{\psi}\gamma^\beta\frac{g}{4\cos\theta_W}(2I_3-q)\psi_L \\ &= -\bar{\psi}\gamma^\beta\frac{g}{4\cos\theta_W}2I_3 + q\bar{\psi}\gamma^\beta\psi_L \end{aligned}$$

The full left-handed interaction therefore becomes

$$\begin{aligned} i\bar{\psi}\not{D}\psi^{12} &= i\bar{\psi}^L\gamma^\beta\partial_\beta\psi - \frac{g}{4}\left(W_\beta^+J_-^\beta + W_\beta^-J_+^\beta\right) - qA_\beta\bar{l}\gamma^\beta l \\ &= i\bar{\psi}^L\gamma^\beta\left(\partial_\beta\psi + \frac{i}{4}g\left(W_\beta^+(\tau_1-i\tau_2)+W_\beta^-(\tau_1+i\tau_2)\right)\psi\right) \\ &\quad + i\bar{\psi}^L\gamma^\beta\frac{i}{4\cos\theta_W}\left(gZ_\beta^0\left(I-\frac{Y_W}{2}\right)+qA_\beta g\right)\psi \end{aligned}$$

where

$$q = I_3 + \frac{Y_W}{2}$$

Expanding the covariant derivatives,

$$\begin{aligned} \mathscr{L}_{EW,leptons} &= \frac{1}{4}F_{\alpha\beta}^iF^{i\alpha\beta} + \frac{1}{4}H_{\alpha\beta}H^{\alpha\beta} \\ &\quad + \sum_{l=e,\mu,\tau} i\bar{\psi}\left(\gamma^\beta\partial_\beta\psi_l^L + \frac{i}{2}g\gamma^\beta B_\beta^j\tau_j\psi_l^L + \frac{i}{2}g'\gamma^\beta B_\beta Y\psi_l^L\right) \\ &\quad + \sum_{l=e,\mu,\tau} i\bar{\psi}\left(\gamma^\beta\partial_\beta\psi_l^R + \frac{i}{2}g'\gamma^\beta B_\beta Y\psi_l^R\right) \\ &\quad + \left|\partial_\beta\phi^a + \frac{i}{2}gB_\beta^j(\tau_j\phi)^a + \frac{i}{2}g'B_\beta Y\phi^a\right|^2 + \lambda\left(|\phi|^2-v^2\right)^2 \\ &\quad + \sum_{l=e,\mu,\tau} G_l\left(\phi^\dagger\bar{\psi}_l^R\psi_l^L + \bar{\psi}_l^L\psi_l^R\phi\right) \end{aligned}$$

Exercise Write out all terms in the Lagrange density involving the electron in detail, expanding the Higgs field about its minimum, $\phi = \begin{pmatrix} v+\rho \\ 0 \end{pmatrix}$, and putting the gauge fields in terms of $W_\alpha^\pm, Z_\alpha^0$ and A_α.

8.4 Weak interactions of quarks

We have written the Lagrange density for quarks as

$$S_{quarks} = \frac{1}{4}\sum_{a=1}^{8}\int G_{\alpha\beta}^aG^{a\alpha\beta}d^4x + \sum_{c=r,g,b}\sum_q\int\bar{\psi}_q(i\not{D}-m)\psi_q$$

where

$$\mathbf{D}\psi_q = \mathbf{d}\psi_q - g_s \mathbf{g}^a \lambda_a \psi_q + electroweak$$

and the obvious thing to write down for the electroweak contribution is to repeat the form we have for the leptons:

$$\begin{aligned}
\mathscr{L}_{EW,quarks} &= \sum_{q=u,c,t} i\bar{\psi}_q^L \left(\gamma^\beta \partial_\beta \psi_q^L + \frac{i}{2} g \gamma^\beta B_\beta^j \tau_j \psi_q^L + \frac{i}{2} g' \gamma^\beta B_\beta Y \psi_q^L \right) \\
&\quad + \sum_{q=u,c,t} i\bar{\psi}_q^R \left(\gamma^\beta \partial_\beta \psi_q^R + \frac{i}{2} g' \gamma^\beta B_\beta Y \psi_q^R \right) \\
&\quad + \sum_{l=e,\mu,\tau} G_l \left(\phi^\dagger \bar{\psi}_q^R \psi_q^L + \bar{\psi}_q^L \psi_q^R \phi \right)
\end{aligned}$$

Notice that this Lagrange density only links quarks in the same electroweak doublet. This means that, for example, the s quark can emit a W^- boson, decaying into a c quark. However, additional decays have been observed, and it is possible for the s quark to decay into *any* of the three positively charged quarks, u, c, t. The solution to this problem is to introduce a *mixing matrix*.

The first introduction of a mixing matrix was introduced by Cabibbo in 1963 to explain the decay of strange particles into non-strange particles. This process was later understood as the weak decay of the s quark into the u quark. Cabibbo suggested that in gauge couplings of the form

$$-\frac{g}{2} \bar{u} \gamma^\beta W_\beta^- \tau_j d$$

the d quark should be replaced by the linear combination

$$d' = d \cos\theta_c + s \sin\theta_c$$

where, to agree with experiment the Cabibbo angle, θ_c, should be about 13.04°. With the discovery of the c quark a decade later, it was found that both the s, and d quarks could decay into either u or c, so the Cabibbo angle was generalized to a mixing matrix,

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

There remains a more subtle problem with this mixing matrix, which was recognized by Kobayashi and Maskawa. In 1973, they proved that the four-quark model could not account for the observed CP violation in weak decays. To solve the problem, they proposed a third pair of quarks, the t, b doublet. The b quark was seen in 1976 and the t in 1995. With the additional quarks, the 3×3 extension of the Cabibbo matrix, the KM matrix, had enough degrees of freedom to allow CP violation.

Cronin and Fitch won the 1980 Nobel prize "for the discovery of violations of fundamental symmetry principles in the decay of neutral K-mesons"; half the 2008 Nobel prize was awarded to Kobayashi and Maskawa "for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature". (The other half went to Nambu, "for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics").

Rather than applying the CKM mixing matrix to the interaction term, it is equally effective to apply it to the Yukawa term. The essential point is that the mass eigenstates must be different than the weak interaction eigenstates. Therefore, it is equally satisfactory to write either

$$\begin{aligned}
\mathscr{L}_{EW,quarks} &= \sum_{q=u,c,t} i\bar{\psi}_q^L \gamma^\beta D_\beta \left(G_{qq'} \psi_{q'}^L \right) \\
&\quad + \sum_{q=u,c,t} i\bar{\psi}_q^R \gamma^\beta D_\beta \psi_q^R \\
&\quad + \sum_{q=u,c,t} G_q \left(\phi^\dagger \bar{\psi}_q^R \psi_q^L + \bar{\psi}_q^L \psi_q^R \phi \right)
\end{aligned}$$

or

$$
\begin{aligned}
\mathscr{L}_{EW,quarks} &= \sum_{q=u,c,t} i\bar{\psi}_q^L \gamma^\beta D_\beta \psi_{q'}^L \\
&\quad + \sum_{q=u,c,t} i\bar{\psi}_q^R \gamma^\beta D_\beta \psi_q^R \\
&\quad + \sum_q \left(\phi^\dagger \bar{\psi}_q^R G_{qq'} \psi_{q'}^L + \bar{\psi}_{q'}^L G_{q'q} \psi_q^R \phi\right)
\end{aligned}
$$

and the second expression is simpler. Giving different masses to different quarks is effectively a coupling between the families of quarks, allowing them to decay into one another. The matrix $G_{qq'}$ is called the Cabibbo-Kobayashi-Maskawa matrix, or simply the CKM matrix

Similar inter-family decays have now been shown to occur between lepton families as well, leading to mixing of the neutrinos. This provides evidence for neutrino masses, so it now seems likely that a similar mixing matrix is required for the lepton Yukawa terms,

$$
\sum_l \left(\phi^\dagger \bar{\psi}_l^R G_{ll'} \psi_{l'}^L + \bar{\psi}_{l'}^L G_{l'l} \psi_l^R \phi\right)
$$

The lepton mixing matrix is called the Pontecorvo-Maki–Nakagawa–Sakata (PMNS) matrix.

For three generations of leptons, the matrix can be written as:

$$
\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}
$$

where

$$
\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}
$$

are the neutrino fields participating in the weak interaction, and

$$
\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}
$$

are the mass eigenstates.

Looking at the Yukawa term, at the vacuum expectation value of the Higgs, we have, for example

$$
v\left(\bar{\nu}_e^R G_{\nu_e e} e + \bar{e}^L G_{e\nu_e} \psi_l^R \phi\right)
$$

9 Putting it all together

Our final Lagrangian for the standard model is, in abbreviated form,

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{4}\sum_{a=1}^{8} G^a_{\alpha\beta} G^{a\alpha\beta} d^4x + \frac{1}{4}\sum_{i=1}^{3} F^i_{\alpha\beta} F^{i\alpha\beta} + \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta} \\
&\quad + \sum_{c=r,g,b} \sum_{q=1}^{3\,doublets} \int \bar{\psi}_q^L i\not{D}\psi_q^L + \sum_{c=r,g,b} \sum_{q=1}^{6\,singlets} \int \bar{\psi}_q^R i\not{D}\psi_q^R \\
&\quad + \sum_{l=e,\mu,\tau}^{3\,doublets} i\bar{\psi}^L \not{D}\psi_l^L + \sum_{l=1}^{6\,singlets} i\bar{\psi}^R \not{D}\psi_l^R \\
&\quad + |D\phi^a|^2 + \lambda\left(|\phi|^2 - v^2\right)^2 \\
&\quad + \sum_q \left(\phi^\dagger \bar{\psi}_q^R G_{qq'} \psi_{q'}^L + \bar{\psi}_{q'}^L G_{q'q} \psi_q^R \phi\right) + \sum_l \left(\phi^\dagger \bar{\psi}_l^R G_{ll'} \psi_{l'}^L + \bar{\psi}_{l'}^L G_{l'l} \psi_l^R \phi\right)
\end{aligned}
$$

Expanding out the covariant derivatives, this becomes

$$\begin{aligned}
\mathscr{L} &= \frac{1}{4}\sum_{a=1}^{8} G^{a}_{\alpha\beta}G^{a\alpha\beta}d^{4}x+\frac{1}{4}\sum_{i=1}^{3} F^{i}_{\alpha\beta}F^{i\alpha\beta}+\frac{1}{4}H_{\alpha\beta}H^{\alpha\beta} \\
&\quad \sum_{c=r,g,b}\sum_{q}^{doublets} i\bar{\psi}^{L}_{q}\left(\gamma^{\beta}\partial_{\beta}\psi^{L}_{q}+\frac{i}{2}g\gamma^{\beta}B^{j}_{\beta}\tau_{j}\psi^{L}_{q}+\frac{i}{2}g'\gamma^{\beta}B_{\beta}Y\psi^{L}_{q}\right) \\
&\quad +\sum_{c=r,g,b}\sum_{q}^{singlets} i\bar{\psi}^{R}_{q}\left(\gamma^{\beta}\partial_{\beta}\psi^{R}_{q}+\frac{i}{2}g'\gamma^{\beta}B_{\beta}Y\psi^{R}_{q}\right) \\
&\quad +\sum_{l=e,\mu,\tau}^{doublets} i\bar{\psi}\left(\gamma^{\beta}\partial_{\beta}\psi^{L}_{l}+\frac{i}{2}g\gamma^{\beta}B^{j}_{\beta}\tau_{j}\psi^{L}_{l}+\frac{i}{2}g'\gamma^{\beta}B_{\beta}Y\psi^{L}_{l}\right) \\
&\quad +\sum_{l}^{singlets} i\bar{\psi}\left(\gamma^{\beta}\partial_{\beta}\psi^{R}_{l}+\frac{i}{2}g'\gamma^{\beta}B_{\beta}Y\psi^{R}_{l}\right) \\
&\quad +\left|\partial_{\beta}\phi^{a}+\frac{i}{2}gB^{j}_{\beta}\left(\tau_{j}\phi\right)^{a}+\frac{i}{2}g'B_{\beta}Y\phi^{a}\right|^{2}+\lambda\left(|\phi|^{2}-v^{2}\right)^{2} \\
&\quad +\sum_{q}\left(\phi^{\dagger}\bar{\psi}^{R}_{q}G_{qq'}\psi^{L}_{q'}+\bar{\psi}^{L}_{q'}G_{q'q}\psi^{R}_{q}\phi\right)+\sum_{l}\left(\phi^{\dagger}\bar{\psi}^{R}_{l}G_{ll'}\psi^{L}_{l'}+\bar{\psi}^{L}_{l'}G_{l'l}\psi^{R}_{l}\phi\right)
\end{aligned}$$

where the gauge field strengths are given by

$$\begin{aligned}
G^{a}_{\alpha\beta} &= \partial_{\alpha}g^{a}_{\beta}-\partial_{\beta}g^{a}_{\alpha}+\frac{1}{2}f_{bc}{}^{a}g^{b}_{\alpha}g^{c}_{\beta} \\
F^{i}_{\alpha\beta} &= \partial_{\alpha}B^{i}_{\beta}-\partial_{\beta}B^{i}_{\alpha}+\frac{1}{2}\varepsilon_{jk}{}^{i}B^{j}_{\alpha}B^{k}_{\beta} \\
H_{\alpha\beta} &= \partial_{\alpha}B_{\beta}-\partial_{\beta}B_{\alpha}
\end{aligned}$$

The left-handed doublets are

$$\begin{aligned}
\psi^{L}_{q} &= \frac{1}{2}(1-\gamma_{5})\begin{pmatrix} u \\ d \end{pmatrix},\frac{1}{2}(1-\gamma_{5})\begin{pmatrix} c \\ s \end{pmatrix},\frac{1}{2}(1-\gamma_{5})\begin{pmatrix} t \\ b \end{pmatrix} \\
\psi^{L}_{l} &= \frac{1}{2}(1-\gamma_{5})\begin{pmatrix} e \\ \nu_{e} \end{pmatrix},\frac{1}{2}(1-\gamma_{5})\begin{pmatrix} \mu \\ \nu_{\mu} \end{pmatrix},\frac{1}{2}(1-\gamma_{5})\begin{pmatrix} \tau \\ \nu_{\tau} \end{pmatrix}
\end{aligned}$$

and the right-handed singlets are $\frac{1}{2}(1+\gamma_{5})$ times

$$\begin{aligned}
\psi_{q} &= u,d,c,s,t,b \\
\psi_{l} &= e,\mu,\tau
\end{aligned}$$

By choosing the $SU(2)$ gauge appropriately, the complex Higgs doublet,

$$\phi=\begin{pmatrix} \phi_{1} \\ \phi_{2} \end{pmatrix}$$

may be given the form

$$\phi=\begin{pmatrix} v+\rho(x) \\ 0 \end{pmatrix}$$

with v a real constant and ρ a real scalar field. When $\phi=\begin{pmatrix} v+\rho(x) \\ 0 \end{pmatrix}$ is substituted into the Lagrangian, the gauge-field combinations

$$\begin{aligned}
W^{+}_{\alpha} &= B^{1}_{\alpha}+iB^{2}_{\alpha} \\
W^{-}_{\alpha} &= B^{1}_{\alpha}-iB^{2}_{\alpha} \\
Z^{0}_{\alpha} &= B^{3}_{\alpha}\cos\theta_{W}-B_{\alpha}\sin\theta_{W}
\end{aligned}$$

acquire masses $m_{W^{\pm}} = \frac{gv}{2}$ and $m_Z = \frac{gv}{2\cos\theta_W}$ where

$$\cos\theta_W = \frac{m_W}{m_{Z_0}}$$

gives the Weinberg angle. The remaining massless combination of B^3_α and B_α is identified with the photon

$$A_\alpha = B^3_\alpha \sin\theta_W + B_\alpha \cos\theta_W$$

10 Flavor symmetry

We now consider the approximate symmetry between different *flavors* of quark. In the simplest version of flavor symmetry, we consider sufficiently high energy to allow us to neglect the mass differences between the u, d and s quarks. Then there is a symmetry, $SU(3)$, that rotates each of these three quark fields into the others. Notice that this is quite distinct from the $SU(3)_{color}$ symmetry that gives rise to the gluons and strong interaction. This is an approximate symmetry, and may be extended to $SU(6)$ by including the c, t and b quarks.

10.1 Standard form of a Lie algebra

The first step in analyzing the particle states under $SU(3)_{flavor}$ is to put the Lie algebra into standard form. The standard form was developed by Cartan as a way of classifying all semisimple Lie groups, but its application to $SU(n)$ is especially easy.

In general, Cartan's standard form divides the generators, G_A, into two subsets, $\{H_i\}, \{E_\alpha\}$ where the H_i form a maximal set of mutually commuting generators,

$$[H_i, H_j] = 0$$

For the special unitary groups, $su(n)$, there are $n-1$ of these. We may choose a basis in which these are diagonal. The remaining generators, E_α, are each chosen to satisfy an equation of the form

$$[H_i, E] = \rho_i E$$

for some constants ρ_i and for each $i = 1, \ldots n-1$. If we start with a general basis, each such equation is an eigenvalue equation, since we may expand

$$\begin{aligned} H &= a^A G_A \\ E &= b^A G_A \end{aligned}$$

Then, expanding the commutation relation,

$$\begin{aligned} a^A b^B [G_A, G_B] &= \rho b^B G_B \\ a^A b^B c_{AB}{}^C &= \rho b^C \end{aligned}$$

and defining

$$M^C_B = a^A c_{AB}{}^C$$

we have the usual form of an eigenvalue equation:

$$M^C_B b^B = \rho b^C$$

The standard form for the full Lie algebra becomes

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_i, E_\alpha] &= \rho_{i\alpha} E_\alpha \\ [E_\alpha, E_\beta] &= \eta_{\alpha\beta} E_{\alpha+\beta} \end{aligned}$$

A few theorems are required to show that this is always possible, but for $SU(n)$ we can demonstrate this form explicitly and will not need the theorems.

The usefulness of this form stems from the relationship between the H_i and the E_α generators. Suppose we find a state labeled by eigenvalues of the H_i,

$$H_i|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$$

Then acting on the eigenstate with any of the E_α we have

$$\begin{aligned} H_i(E_\alpha|\lambda_i\rangle) &= (E_\alpha H_i + \rho_{i\alpha}E_\alpha)|\lambda_i\rangle \\ &= (\lambda_i+\rho_{i\alpha})(E_\alpha|\lambda_i\rangle) \end{aligned}$$

and we have a new eigenket with eigenvalue $\lambda_i + \rho_{i\alpha}$. By applying all of the E_α sufficiently many times, we may generate a complete set of eigenstates. The application of this method to $SU(2)$ is used in quantum mechanics to generate all possible representations of angular momentum.

10.2 Standard form for su(2)

To show how the method works, we first look at $SU(2)$. In this case, there is only one diagonal operator,

$$\begin{aligned} H &= \frac{1}{2}\sigma_3 \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

The remaining two generators, σ_1, σ_2 are combined as

$$E_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$$

These become nilpotent matrices,

$$\begin{aligned} E_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ E_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

The Lie algebra becomes,

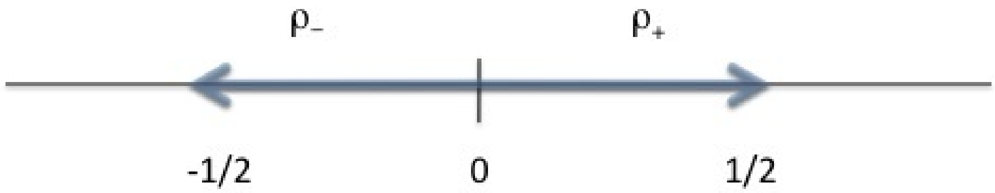
$$\begin{aligned} [H,H] &= 0 \\ [H,E_\pm] &= \pm\frac{1}{2}E_\pm \\ [E_+,E_-] &= 2H \end{aligned}$$

which is indeed of standard form.

The next step is to think of the collection of diagonal operators, H_i, as a vector, $\mathbf{H}$, in an $n-1$ dimensional space. Then the eigenvalues, $\rho_{i\alpha}$ for each α also form a vector, ρ_α. For $SU(2)$ these are the 1-dimensional vectors,

$$\begin{aligned} \mathbf{H} &= (H) \\ \rho_+ &= \left(\frac{1}{2}\right) \\ \rho_- &= \left(-\frac{1}{2}\right) \end{aligned}$$

and we may plot them:



This plot gives us a graphical representation of the possible states, namely, the two eigenstates of the fundamental representation. If we label these states by ρ and the eigenvalue of the Casimir operator $\mathbf{J}^2$, then we have the usual quantum mechanical notation,

$$|j,m\rangle$$

where $m = \pm\frac{1}{2}$. To form states with higher values of angular momentum, we take products of spin-$\frac{1}{2}$ states, and using the techniques of Young tableau or of Clebsch-Gordon reduction, re-express them as irreducible states. This procedure gives us $2j+1$ states $|j,m\rangle$ for each positive half-integer, j, and all $-j \le m \le j$.

10.3 Standard form for su(3)

Among the eight generators of $SU(3)$, we may diagonalize two. Starting with the Gell-Mann matrices as our basis, the diagonal generators are

$$\begin{aligned}
H_1 = \frac{1}{\sqrt{6}}\lambda_3 &= \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
H_2 = \frac{1}{\sqrt{6}}\lambda_8 &= \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}$$

The normalization is chosen so that the weight diagrams have equally spaced weights. Treat the two diagonal generators as a vector, $\mathbf{H} = (H_1, H_2)$. The remaining generators fall into pairs,

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}
\end{aligned}$$

and we immediately see that we can take

$$\begin{aligned}
E_1^{\pm} &= \frac{1}{2}(\lambda_1 \pm i\lambda_2) \\
E_2^{\pm} &= \frac{1}{2}(\lambda_4 \pm i\lambda_5) \\
E_3^{\pm} &= \frac{1}{2}(\lambda_6 \pm i\lambda_7)
\end{aligned}$$

Each generator has a 1 in one off-diagonal position, and zeros everywhere else, for example,

$$E_1^{+} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since H_1, H_2 are diagonal, we may identify particle states with the similtaneous eigenstates:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

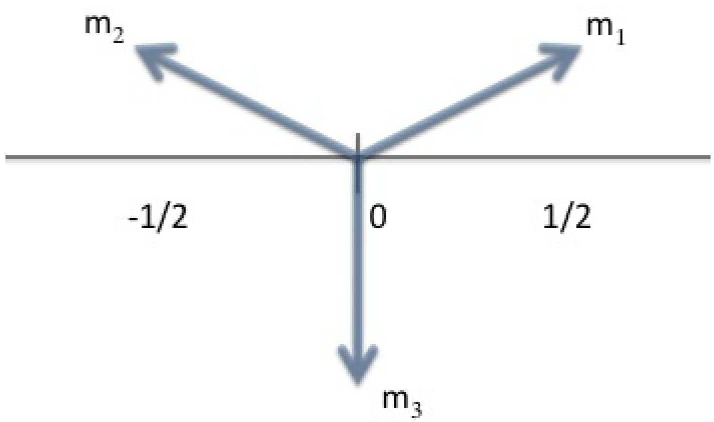
Writing the eigenvalues as

$$\mathbf{H} u_\alpha = \mathbf{m}_\alpha u_\alpha$$

We may immediately write

$$\begin{aligned}
\mathbf{m}_1 &= \left(\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) \\
\mathbf{m}_2 &= \left(-\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) \\
\mathbf{m}_3 &= \left(0, -\frac{2}{3\sqrt{2}} \right)
\end{aligned}$$

and plot



This is the first fundamental representation of $SU(3)$. There is a second fundamental representation found by antisymmetrizing outer products of the generators, $\bar{u}_k = \frac{1}{\sqrt{2}} \varepsilon_{ijk} u_i u_j$ with eigenvalues

$$\begin{aligned}
\mathbf{H} \bar{u}_1 &= \bar{\mathbf{m}}_1 \bar{u}_1 \\
&= \mathbf{H}(u_2 u_3) \\
&= ((\mathbf{H} u_2) u_3 + u_2 \mathbf{H} u_3) \\
&= (\mathbf{m}_2 + \mathbf{m}_3) u_2 u_3 \\
&= \left(\left(-\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) + \left(0, -\frac{2}{3\sqrt{2}} \right) \right) u_2 u_3 \\
&= \left(-\frac{1}{\sqrt{6}}, -\frac{1}{3\sqrt{2}} \right) \bar{u}_1 \\
\mathbf{H} \bar{u}_2 &= (\mathbf{m}_3 + \mathbf{m}_1) \bar{u}_2 \\
&= \left(\frac{1}{\sqrt{6}}, -\frac{1}{3\sqrt{2}} \right) \bar{u}_2 \\
\mathbf{H} \bar{u}_3 &= (\mathbf{m}_1 + \mathbf{m}_2) \bar{u}_3 \\
&= \left(0, \frac{2}{3\sqrt{2}} \right) \bar{u}_3
\end{aligned}$$

so the weight diagram is the same as the first fundamental representation reflected through the H_1 axis, and $\bar{\mathbf{m}}_i = -\mathbf{m}_i$.

10.4 Representations of SU(3)

We may now construct general representations of $SU(3)$ by building tensors from the eigenvectors. The eigenvalues of $\mathbf{H}$ for tensor products are additive, since we have

$$\begin{aligned}
\mathbf{H}(u_i u_j) &= (\mathbf{H}\otimes 1 + 1\otimes \mathbf{H})(u_i\otimes u_j) \\
&= \mathbf{H}u_i\otimes u_j + u_i\otimes \mathbf{H}u_j \\
&= \mathbf{m}_i u_i\otimes u_j + u_i\otimes \mathbf{m}_j u_j \\
&= (\mathbf{m}_i + \mathbf{m}_j)\, u_i\otimes u_j
\end{aligned}$$

We can find irreducible representations by acting with the various $E^{\pm}_{\alpha}$, since these move states only within irreducible representations.

10.4.1 Meson octet

The next simplest representation is the product of the two fundamental representations,

$$3\otimes\bar{3} = 1\oplus 8$$

The states are built from the products $u_i\bar{u}_j$, which have eigenvalues as follows:

$$\begin{aligned}
\mathbf{H}u_1\bar{u}_1 &= (0,0) \\
\mathbf{H}u_1\bar{u}_2 &= \left(\frac{2}{\sqrt{6}},0\right) \\
\mathbf{H}u_1\bar{u}_3 &= \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{2}}\right) \\
\mathbf{H}u_2\bar{u}_1 &= \left(-\frac{2}{\sqrt{6}},0\right) \\
\mathbf{H}u_2\bar{u}_2 &= (0,0) \\
\mathbf{H}u_2\bar{u}_3 &= \left(-\frac{1}{\sqrt{6}},\frac{1}{\sqrt{2}}\right) \\
\mathbf{H}u_3\bar{u}_1 &= \left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{2}}\right) \\
\mathbf{H}u_3\bar{u}_2 &= \left(\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{2}}\right) \\
\mathbf{H}u_3\bar{u}_3 &= (0,0)
\end{aligned}$$

There are three degenerate states with quantum numbers $(0,0)$. Certain linear combinations of these correspond to definite particle states; the remaining combinations refer to definite particles. To help identify the particles, note that the electric charge for the mesons is given by $Q = I_3 + \frac{1}{2}Y_s$, where the isospin, I_3, and strong hypercharge, Y_s, are related to the eigenvalues of $\mathbf{H}$ by

$$(I_3, Y_s) = \left(\frac{\sqrt{6}}{2}m_1, \sqrt{2}m_2\right)$$

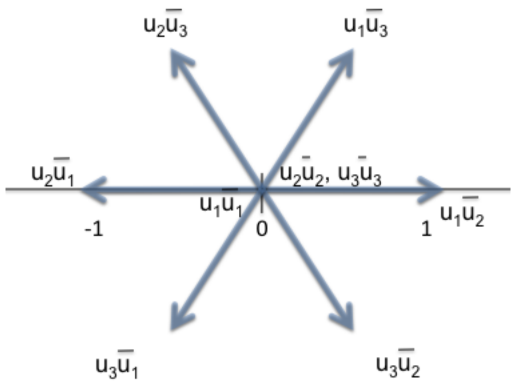
Therefore,

$$Q = \frac{\sqrt{6}}{2}m_1 + \frac{1}{\sqrt{2}}m_2$$

The charges of the various parings are therefore,

$$\begin{aligned}
Q_{u_1\bar{u}_1} &= 0 \\
Q_{u_1\bar{u}_2} &= 1 \\
Q_{u_1\bar{u}_3} &= 1 \\
Q_{u_2\bar{u}_1} &= -1 \\
Q_{u_2\bar{u}_2} &= 0 \\
Q_{u_2\bar{u}_3} &= 0 \\
Q_{u_3\bar{u}_1} &= -1 \\
Q_{u_3\bar{u}_2} &= 0 \\
Q_{u_3\bar{u}_3} &= 0
\end{aligned}$$

Now plot the states:



Now we can identify the quarks. The center row contains a triplet built from u_1, u_2 and their antiparticles, having charges $-, 0, +$. We identify these with the pions:

$$\begin{aligned}
\pi^- &= u_2\bar{u}_1 \\
\pi^0 &= \frac{1}{\sqrt{2}}(u_1\bar{u}_1 + u_2\bar{u}_2) \\
\pi^+ &= u_1\bar{u}_2
\end{aligned}$$

and since the pion has strangeness 0, we identify the constituent quarks as

$$\begin{aligned}
u &= u_1 \\
\bar{u} &= \bar{u}_1 \\
d &= u_2 \\
\bar{d} &= \bar{u}_2
\end{aligned}$$

Then the pions are then identified with the quark-antiquark combinations

$$\begin{aligned}
\pi^+ &= u\bar{d} \\
\pi^0 &= \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right) \\
\pi^- &= d\bar{u}
\end{aligned}$$

We may also identify the third quark as $u_3 = s$, and complete the identification of the meson octet:

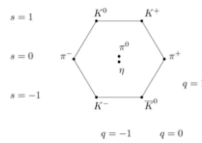
$$\begin{aligned}
K^0 &= d\bar{s} \\
K^+ &= u\bar{s} \\
K^- &= s\bar{u} \\
\bar{K}^0 &= s\bar{d}
\end{aligned}$$

The remaining state of the meson octet, η, and the singlet state, η', show some mixing due to differences in the quark masses.

$$\eta, \eta' \sim \alpha u\bar{u} + \beta d\bar{d} + \gamma s\bar{s}$$

Because of the mixing, this is sometimes called the meson nonet.

We will not venture into the details of the central degeneracy, but there is a good review on the Particle Data Group website. The final octet is



10.4.2 Baryon singlet and decuplet

Analysis with Young tableau shows that the product three 3-quark states has irreducible representations,

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

The simplest representation is the singlet

$$u_{[i} u_j u_{k]} = \varepsilon_{ijk} u_1 u_2 u_3$$

with

$$\begin{aligned}
\mathbf{H} u_1 u_2 u_3 &= (\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3) u_1 u_2 u_3 \\
&= \left(\left(\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) + \left(-\frac{1}{\sqrt{6}}, \frac{1}{3\sqrt{2}} \right) + \left(0, -\frac{2}{3\sqrt{2}} \right) \right) u_1 u_2 u_3 \\
&= 0
\end{aligned}$$

This state is called the Λ, but the ground state is forbidden by the requirement of total antisymmetry of the wave function. There have been recent claims that interpret some higher energy states as excited states of the Λ, though the data is inconculsive.

We were able to guess the contents of the meson octet, but with 3 quarks it pays to be a little more systematic. We approach the problem in the spirit of Clebsch-Gordon coefficients, starting with the highest state of a given symmetry, and using lowering operators to trace out the remaining states.

One of the maximal symmetry states we can write consists of 3 up quarks in a totally symmetric combination. The isospins, m_u, will line up, leading to an $I_3 = \frac{3}{2}$ state. The Y_s values also add, so we have

$$\mathbf{H}(uuu) = \left(\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{2}} \right)(uuu)$$

The charge of this state is given by

$$\begin{aligned}
Q &= \frac{\sqrt{6}}{2} m_1 + \frac{1}{\sqrt{2}} m_2 \\
&= \frac{3}{2} + \frac{1}{2} \\
&= +2
\end{aligned}$$

which is a bit surprising, but turns out to correspond to a particle called the Δ^{++}.

To proceed, we need to know how E_α^- changes the eigenvalues, for $\alpha = 1, 2, 3$. This follows from the commutation relations

$$
\begin{aligned}
\left[H_1, E_1^{\pm}\right] &= \pm\frac{2}{\sqrt{6}}E_1^{\pm} \\
\left[H_1, E_2^{\pm}\right] &= \pm\frac{1}{\sqrt{6}}E_2^{\pm} \\
\left[H_1, E_3^{\pm}\right] &= \mp\frac{1}{\sqrt{6}}E_3^{\pm} \\
\left[H_2, E_1^{\pm}\right] &= 0 \\
\left[H_2, E_2^{\pm}\right] &= \pm\frac{1}{\sqrt{2}}E_2^{\pm} \\
\left[H_2, E_3^{\pm}\right] &= \pm\frac{1}{\sqrt{2}}E_3^{\pm}
\end{aligned}
$$

Consider application of E_1^- to the uuu state. Since

$$
\left[\mathbf{H}, E_1^-\right] = -\left(\frac{2}{\sqrt{6}}, 0\right)E_1^-
$$

we know that whenever $\mathbf{H}\left|\mathbf{m}\right\rangle = \mathbf{m}\left|\mathbf{m}\right\rangle$ that

$$
\begin{aligned}
\mathbf{H}\left(E_1^-\left|\mathbf{m}\right\rangle\right) &= \left(E_1^-\mathbf{H} - \left(\frac{2}{\sqrt{6}}, 0\right)E_1^-\right)\left|\mathbf{m}\right\rangle \\
&= \left(\mathbf{m} - \left(\frac{2}{\sqrt{6}}, 0\right)\right)E_1^-\left|\mathbf{m}\right\rangle
\end{aligned}
$$

and we may identify

$$
E_1^-\left|\mathbf{m}\right\rangle = c\left|m_1 - \frac{2}{\sqrt{6}}, m_2\right\rangle
$$

Applying this reasoning to the uuu state, and noting that

$$
\begin{aligned}
E_1^- u &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= d
\end{aligned}
$$

while

$$
E_1^- d = 0
$$

we have

$$
E_1^-\left(uuu\right) = c\left(duu + udu + uud\right)
$$

and normalization requires $c = \frac{1}{\sqrt{3}}$. Acting with H_1, H_2 shifts the eigenvalues by $\left(-\frac{2}{\sqrt{6}}, 0\right)$, so the new state has $\mathbf{m} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$. Applying E_1^- two more times yields two additional states

$$
\begin{gathered}
\tfrac{1}{\sqrt{3}}\left(ddu + dud + udd\right) \\
ddd
\end{gathered}
$$

with eigenvalues $\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$. Using

$$
Q = \frac{\sqrt{6}}{2}m_1 + \frac{1}{\sqrt{2}}m_2
$$

as for the mesons, this quadruplet of particles has charges $+2,+1,0,-1$, and we have the Ω baryons,

$$\begin{aligned}
\Delta^{++} &= uuu \\
\Delta^{+} &= \frac{1}{\sqrt{3}}(duu+udu+uud) \\
\Delta^{0} &= \frac{1}{\sqrt{3}}(ddu+dud+udd) \\
\Delta^{-} &= ddd
\end{aligned}$$

However, we have not yet exhausted the particle states of the totally symmetric decuplet.

Now we start again with the Ω^{++} state, and apply E_2^-, which changes the eigenvalues by ρ_2 given by

$$\left[\mathbf{H}, E_2^-\right] = \rho_2 E_2^-$$

with

$$\rho_2 = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}\right)$$

From the states $\Delta^{++} = \left(\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$ we have

$$\begin{aligned}
E_2^-\Delta^{++} &= \left(\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}\right) \\
&= \left(\frac{2}{\sqrt{6}}, 0\right) \\
\left(E_2^-\right)^2\Delta^{++} &= \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}\right) \\
\left(E_2^-\right)^3\Delta^{++} &= \left(0, -\sqrt{2}\right)
\end{aligned}$$

with electric charges $Q = \frac{\sqrt{6}}{2}m_1 + \frac{1}{\sqrt{2}}m_2 = (2,1,0,-1)$, respectively. The quark content is found by noting that

$$\begin{aligned}
E_2^- u &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= s \\
E_2^- d &= E_2^- s = 0
\end{aligned}$$

We identify the resulting states as

$$\begin{aligned}
\Delta^{++} &= uuu \\
\Sigma^{*+} &= \frac{1}{\sqrt{3}}(uus+usu+suu) \\
\Xi^{0} &= \frac{1}{\sqrt{3}}(uss+sus+ssu) \\
\Omega^{-} &= sss
\end{aligned}$$

We may act with E_1^- on each of these. Recalling $E_1^- u = d$ and noting that $E_1^- d = E_2^- s = 0$, we find three additional states:

$$\begin{aligned}
\Sigma^{*0} &= \frac{1}{\sqrt{6}}(dus+dsu+uds+sdu+usd+sud) \\
\Sigma^{*-} &= \frac{1}{\sqrt{3}}(dds+dsd+dds) \\
\Xi^{-} &= \frac{1}{\sqrt{3}}(dss+sds+ssd)
\end{aligned}$$

10.4.3 Baryon octets

Now we turn to the octets. The two octets are built from the same sets of quarks but with different angular momentum, $j = \frac{1}{2}, \frac{3}{2}$, and are therefore considered to describe different energy levels of the same quark-composite particle. We consider the ground state particles, with $j = \frac{1}{2}$. Finding the states works just like finding Clebsch-Gordon coefficients. The Δ^{++} is uniquely determined, but

$$\Delta^{+} = \frac{1}{\sqrt{3}} (uud + udu + duu)$$

is only one of two states that can be built from two up and one down quark. The orthogonal state is

$$N^{+} = \frac{1}{\sqrt{6}} (uud - 2udu + duu)$$

This state has $Q = 1$ and vanishing strangeness, $S = 0$; it is identified with the proton. Note that there are only two independent combinations since the totally antisymmetric part of uud vanishes because there are two u quarks.

Now we apply $E_1^{\pm}$ and E_2^{-} to find the remaining states. The neutron is given by $E_1^{-} p$,

$$N^{0} = \frac{1}{\sqrt{6}} (2dud - ddu - udd)$$

Now apply E_2^{-} $(u \to s)$ to the neutron to find

$$\Sigma^{-} = \frac{1}{\sqrt{6}} (2dsd - dds - sdd)$$

and then two applications of E_1^{+} $(d \to u)$ to get the remaining Sigma baryons,

$$\begin{aligned} \Sigma^{0} &= \frac{1}{2\sqrt{3}} (2usd + 2dsu - uds - dus - sud - sdu) \\ \Sigma^{+} &= \frac{1}{\sqrt{6}} (2usu - uus - suu) \end{aligned}$$

The final states are given by applying E_2^{-} to each of these,

$$\begin{aligned} \Xi^{-} &= E_2^{-} \Sigma^{0} \\ &= \frac{1}{\sqrt{6}} (ssd + dss - 2sds) \\ \Xi^{0} &= E_2^{-} \Sigma^{+} \\ &= \frac{1}{\sqrt{6}} (ssu + uss - 2sus) \end{aligned}$$

There remains one state orthogonal to both the Σ^{0} and the Σ^{*0}. Set

$$\Lambda = \alpha usd + \beta sdu + \gamma dus + \delta uds + \varepsilon dsu + \sigma sud$$

Then orthogonality with Σ^{0} requires

$$2\alpha + 2\varepsilon - \delta - \gamma - \sigma - \beta = 0$$

while orthogonality with Σ^{*0} gives

$$\alpha + \varepsilon + \delta + \gamma + \sigma + \beta = 0$$

Setting $\alpha = -\varepsilon$ and $\sigma = \beta = -\gamma = -\delta$, the Λ becomes

$$\Lambda^{0} = \alpha (usd - dsu) + \beta (sdu + dus - uds - sud)$$

The remaining arbitrariness is because the totally antisymmetric part of Λ^0 must vanish:

$$\begin{aligned}
0 &= \alpha\left((usd-dsu)+(dus-sud)+(sdu-uds)\right) \\
&\quad +\alpha\left(-(sud-dus)-(dsu-usd)-(uds-sdu)\right) \\
&\quad +\beta\left(sdu+dus-uds-sud\right) \\
&\quad +\beta\left(usd+sdu-dsu-uds\right) \\
&\quad +\beta\left(dus+usd-sud-dsu\right) \\
&\quad -\beta\left(uds+dsu-sdu-usd\right) \\
&\quad -\beta\left(dsu+sud-usd-dus\right) \\
&\quad -\beta\left(sud+uds-dus-sdu\right) \\
&= (2\alpha+4\beta)\left(usd+dus+sdu-dsu-sud-uds\right)
\end{aligned}$$

Therefore, set $\beta = -\frac{1}{2}\alpha$ and normalize,

$$\Lambda^0 = \frac{1}{2\sqrt{3}}\left(2usd-2dsu-sdu-dus+uds+sud\right)$$

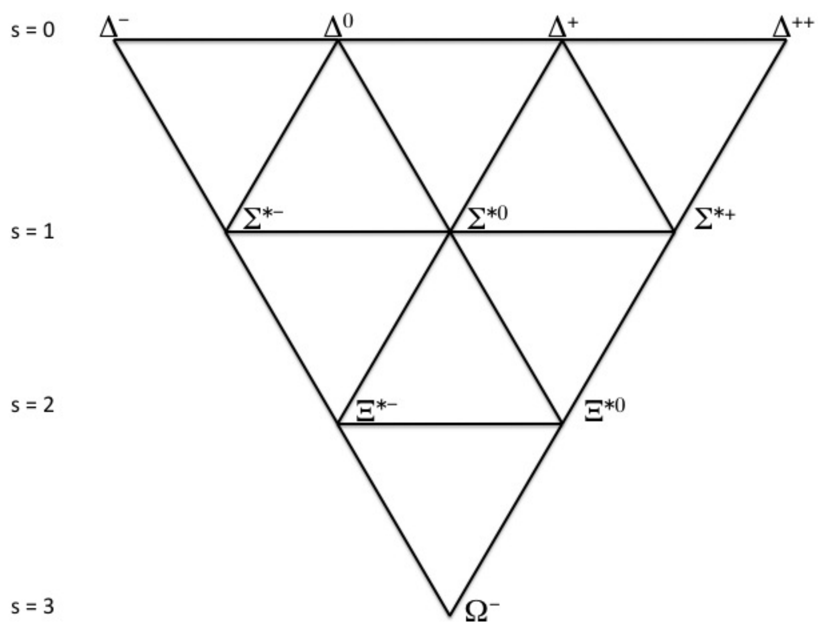
Once again, the singlet state will mix with the Λ^0, Σ^0 and Σ^{*0} because of inequalities in the quark masses.

The strong hypercharge, Y_s, is typically written in terms of more familiar quantum numbers which are conserved by the strong interaction,

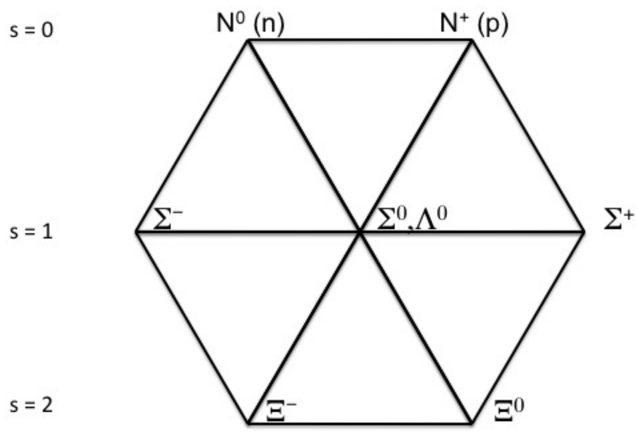
$$Y_s = B+S+C+T+B'$$

Here, B is the baryon number, with $B=\frac{1}{3}$ for each quark. S, C, T and B' are the strangeness, charm, topness and bottomness.

To conclude, we plot the decuplet and the octet.



The octet:



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