Dynamics in special relativity

January 13, 2015

We begin our discussion of special relativity with a power point presentation, available on the website.

1 Spacetime

From the power point presentation, you know that spacetime is a four dimensional vector space vector length

$$\begin{aligned} s^2 &= -c^2 (\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ c^2\tau^2 &= c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \end{aligned}$$

where $\Delta t = t_2 - t_1, \Delta x = x_2 - x_1, \Delta y = y_2 - y_1, \Delta z = z_2 - z_1$ are the coordinate differences of the events at the ends of the vector. We may also write this length for infinitesimal proper time $d\tau$ and infinitesimal proper length ds along an infinitesimal interval,

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

and these are agreed upon by all observers. The latter form allows us to find the proper time or length of an arbitrary curve in spacetime by integrating $d\tau$ or ds along the curve.

The elapsed physical time experienced traveling along any timelike curve is given by integrating $d\tau$ along that curve. Similarly, the proper distance along any spacelike curve is found by integrating ds.

The set of points at zero proper interval, $s^2 = 0$, from a given point, P, is the light cone of that point. The light cone divides spacetime into regions. Points lying inside the light cone and having later time than P lie in the *future* of P. Points inside the cone with earlier times lie in the *past* of P. Points outside the cone are called *elsewhere*.

Timelike vectors from P connect P to past or future points. Timelike curves, $x^\alpha(\lambda)$, are curves whose tangent vector $\frac{dx^\alpha}{d\lambda}(\lambda)$ at any point are timelike vectors, while spacelike curves have tangents lying outside the lightcones of their points. The elapsed physical time experienced traveling along any timelike curve is given by integrating $d\tau$ along that curve. Similarly, the proper distance along any spacelike curve is found by integrating ds.

We refer to the coordinates of an event in spacetime using the four coordinates

$$x^\alpha = (ct, x, y, z)$$

where $\alpha = 0, 1, 2, 3$. We may also write x^α in any of the following ways:

$$\begin{aligned} x^\alpha &= (ct, \mathbf{x}) \\ &= \left(ct, x^i\right) \\ &= \left(x^0, x^i\right) \end{aligned}$$

where $i = 1, 2, 3$. This neatly separates the familiar three spatial components of the vector x^α from the time component, allowing us to recognize familiar relations from non-relativistic mechanics.

The invariant interval allows us to define a metric,

$$\eta_{\alpha\beta} \equiv \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so that we may write the invariant interval as

$$s^2 = \eta_{\alpha\beta} x^\alpha x^\beta$$

or infinitesimally

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

where our use of the Einstein summation convention tells us that these expressions are summed over $\alpha, \beta = 0, 1, 2, 3$. A Lorentz transformation may be defined as any transformation of the coordinates that leaves the length-squared s^2 unchanged. It follows that a linear transformation,

$$y^\alpha = \Lambda^\alpha{}_\beta x^\beta$$

is a Lorentz transformation if and only if

$$\eta_{\alpha\beta} x^\alpha x^\beta = \eta_{\alpha\beta} y^\alpha y^\beta$$

Substituting for y^α and equating the coefficients of the arbitrary symmetric $x^\alpha x^\beta$ we have

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \tag{1}$$

as the necessary and sufficient condition for $\Lambda^\beta{}_\nu$ to be a Lorentz transformation.

2 Relativistic dynamics

2.1 Curves

We now turn to look at motion in spacetime. Consider a particle moving along a curve in spacetime. We can write that curve parametrically by giving each coordinate as a function of some parameter λ:

$$x^\alpha = x^\alpha(\lambda)$$

Such a path is called the *world line* of the particle. Here λ can be any parameter that increases monotonically along the curve. We note two particularly convenient choices for λ. First, we may use the time coordinate, t, in relative to our frame of reference. In this case, we have

$$x^\alpha(t) = (ct, x(t), y(t), z(t))$$

As we shall see, the proper time τ experienced by the particle is often a better choice. Then we have

$$x^\alpha(\tau) = (ct(\tau), x(\tau), y(\tau), z(\tau))$$

The proper time is an excellent choice because it provides the same parameterization in *any* frame of reference.

To calculate the proper time experienced along the world line of the particle between events A and B, just add up the infinitesimal displacements $d\tau$ along the path. Thus

$$\begin{aligned}
\tau_{AB} &= \int_A^B d\tau \\
&= \int_A^B \sqrt{dt^2 - \frac{1}{c^2}\left(dx^i\right)^2} \\
&= \int_{t_A}^{t_B} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{dx^i}{dt}\right)^2} \\
&= \int_{t_A}^{t_B} dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}
\end{aligned}$$

where $\mathbf{v}^2$ is the usual squared magnitude of the 3-velocity. Notice that if $\mathbf{v}^2$ is ever different from zero, then τ_{AB} is *smaller* than the time difference $t_B - t_A$:

$$\tau_{AB} = \int_{t_A}^{t_B} dt \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} \leq \int_{t_A}^{t_B} dt = t_B - t_A$$

Equality holds only if the particle remains at rest in the given frame of reference. This difference has been measured to high accuracy. One excellent test is to study the number of muons reaching the surface of the earth after being formed by cosmic ray impacts on the top of the atmosphere. These particles have a halflife on the order of 10^{-11} seconds, so they would normally travel only a few centimeters before decaying. However, because they are produced in a high energy collision that leaves them travelling toward the ground at nearly the speed of light, many of them are detected at the surface of the earth.

In order to discuss dynamics in spacetime, we need a clearer definition of a spacetime vector. So far, we have taken a vector to be the directed segment between two spacetime events. This has a useful consequence. As we have seen, the components of the vectors to the endpoints of such segments transform via Lorentz transformations, $y^\alpha = \Lambda^\alpha{}_\beta x^\beta$. We now generalize from a vector as a directed spacetime interval to define a 4-vector as *any* set of four functions, $v^\alpha = (a, b, c, d)$, as long as in any other frame of reference the components $\tilde{v}^\alpha = \left(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\right)$ are given by

$$\tilde{v}^\alpha = \Lambda^\alpha{}_\beta v^\beta \tag{2}$$

This generalization is important because we are interested in velocities, forces and other physical quantities which are not simply spacetime intervals.

Identifying objects which transform in the same way as the coordinates allows us to generalize the invariance of the interval to the invariance of 4-vector length,

$$\eta_{\alpha\beta} v^\alpha v^\beta = \eta_{\alpha\beta} \tilde{v}^\alpha \tilde{v}^\beta$$

which follows by combining our new definition of a Lorentz transformation, eq.(1), with the transformation property of v^α, eq.(2).

2.2 The 4-velocity

We next define the 4-velocity of a particle, i.e., a 4-vector that characterizes the velocity. We can get the direction of the particle's motion in spacetime by looking at the tangent vector to the curve,

$$t^\alpha = \frac{dx^\alpha(\lambda)}{d\lambda}$$

We can see that this tangent vector is closely related to the ordinary 3-velocity of the particle by expanding with the chain rule,

$$\begin{aligned} t^\alpha &= \frac{dx^\alpha(\lambda)}{d\lambda} \\ &= \frac{dt}{d\lambda}\frac{dx^\alpha}{dt} \\ &= \frac{dt}{d\lambda}\frac{d}{dt}\left(ct, x^i\right) \\ &= \frac{dt}{d\lambda}\left(c, v^i\right) \end{aligned}$$

where v^i is the usual Newtonian 3-velocity. This is close to what we need, but since λ is arbitrary, so is $\frac{dt}{d\lambda}$. This means that t^α may or may not be a vector. For example, suppose, in some frame of reference, we have chosen coordinate time t as the parameter. Then in that frame of reference, we have

$$\begin{aligned} t^\alpha &= \frac{dx^\alpha(t)}{dt} \\ &= \frac{d}{dt}\left(ct, x(t), y(t), z(t)\right) \\ &= \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\ &= (c, \mathbf{v}) \end{aligned}$$

This is not a vector by our definition, because when we transform to a new frame of reference, $\tilde{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta$, we have

$$\begin{aligned} \tilde{t}^\alpha &= \frac{d\tilde{x}^\alpha}{d\tilde{t}} \\ & \frac{d\left(\Lambda^\alpha{}_\beta x^\beta\right)}{\frac{1}{c}d\left(\Lambda^0{}_\mu x^\mu\right)} \\ &= \frac{\Lambda^\alpha{}_\beta dx^\beta}{\frac{1}{c}\Lambda^0{}_\mu dx^\mu} \\ &= \frac{\Lambda^\alpha{}_\beta dx^\beta}{\frac{1}{c}\left(\Lambda^0{}_0 dx^0 + \Lambda^0{}_i dx^i\right)} \\ &= \frac{\Lambda^\alpha{}_\beta dx^\beta}{\Lambda^0{}_0 dt + \frac{1}{c}\Lambda^0{}_i dx^i} \\ &= \frac{1}{\Lambda^0{}_0 + \frac{1}{c}\Lambda^0{}_i v^i}\Lambda^\alpha{}_\beta \frac{dx^\beta}{dt} \end{aligned}$$

so we get an extraneous factor of $\left(\Lambda^0{}_0 + \frac{1}{c}\Lambda^0{}_i v^i\right)^{-1}$.

We can define a true 4-vector by using the proper time as the parameter. Let the world line be parameterized by the elapsed proper time, τ, of the particle. Then define the *4-velocity,*

$$u^\alpha = \frac{dx^\alpha(\tau)}{d\tau}$$

Since $\tau = \tilde{\tau}$ for all observers, we immediately have

$$\tilde{u}^\alpha = \frac{d\tilde{x}^\alpha(\tau)}{d\tilde{\tau}}$$

$$
\begin{aligned}
&= \frac{d\left(\Lambda^{\alpha}{}_{\beta} x^{\beta}\right)}{d\tau} \\
&= \Lambda^{\alpha}{}_{\beta} \frac{dx^{\beta}}{d\tau} \\
&= \Lambda^{\alpha}{}_{\beta} u^{\beta}
\end{aligned}
$$

and u^{α} is a 4-vector.

A very convenient form for the 4-velocity is given by our expansion of the tangent vector. Just as for general λ, we have

$$u^{\alpha} = \frac{dt}{d\tau}\left(c, v^{i}\right)$$

but now we know what the function in front is. Compute

$$
\begin{aligned}
d\tau &= \sqrt{dt^{2} - \frac{1}{c^{2}}\left(dx^{i}\right)^{2}} \\
&= dt\sqrt{1 - \frac{\mathbf{v}^{2}}{c^{2}}}
\end{aligned}
$$

Then we see that

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^{2}}{c^{2}}}} = \gamma$$

Therefore,

$$u^{\alpha} = \gamma\left(c, v^{i}\right) \tag{3}$$

This is an extremely useful form for the 4-velocity. It is used frequently.

Since u^{α} is a 4-vector, its magnitude

$$\eta_{\alpha\beta} u^{\alpha} u^{\beta}$$

must be invariant! This means that the velocity of every particle in spacetime has the same particular value. Let's compute it:

$$
\begin{aligned}
\eta_{\alpha\beta} u^{\alpha} u^{\beta} &= -\left(u^{0}\right)^{2} + \sum_{i}\left(u^{i}\right)^{2} \\
&= -\gamma^{2} c^{2} + \gamma^{2} \mathbf{v}^{2} \\
&= \frac{-c^{2} + \mathbf{v}^{2}}{1 - \frac{\mathbf{v}^{2}}{c^{2}}} \\
&= -c^{2}
\end{aligned}
$$

This is indeed invariant! Our formalism is doing what it is supposed to do.

Now let's look at how the 4-velocity is related to the usual 3-velocity. If $\mathbf{v}^{2} \ll c^{2}$, the components of the 4-velocity are just

$$u^{\alpha} = \gamma\left(c, v^{i}\right) \approx \left(c, v^{i}\right) \tag{4}$$

The speed of light, c, is just a constant, and the spatial components reduce to precisely the Newtonian velocity. This is just right. Moreover, it takes no new information to write the general form of u^{α} once we know v^{i} – there is no new information, just a different form.

2.3 Energy and momentum

From the 4-velocity it is natural to define the 4-momentum by multiplying by the mass,

$$p^\alpha = mu^\alpha$$

In order for the 4-momentum to be a vector, we require $\tilde{p}^\alpha = \Lambda^\alpha{}_\beta p^\beta$. Since u^α is itself a vector, we have

$$\begin{aligned}\tilde{p}^\alpha &= \tilde{m}\tilde{u}^\alpha \\ &= \tilde{m}\Lambda^\alpha{}_\beta u^\beta\end{aligned}$$

Since this must equal $\Lambda^\alpha{}_\beta p^\beta$, the mass of a particle is *invariant,*

$$\tilde{m} = m$$

As we might expect, the 3-momentum part of p^α is closely related to the Newtonian expression mv^i. In general it is

$$p^i = \frac{mv^i}{\sqrt{1-\frac{v^2}{c^2}}}$$

If $v \ll c$ we may expand the denominator to get

$$\begin{aligned}p^i &\approx mv^i\left(1+\frac{1}{2}\frac{v^2}{c^2}+\cdots\right) \\ &\approx mv^i\end{aligned}$$

Thus, while relativistic momentum differs from Newtonian momentum, they only differ at order $\frac{v^2}{c^2}$. Even for the 7 $mi/\sec$ velocity of a spacecraft which escapes Earth's gravity this ratio is only

$$\frac{v^2}{c^2} = 1.4\times10^{-9}$$

so the Newtonian momentum is correct to parts per billion. In particle accelerators, however, where near-light speeds are commonplace, the difference is substantial (see exercises).

Now consider the remaining component of the 4-momentum. Multiplying by c and expanding γ we find

$$\begin{aligned}p^0c &= mc^2\gamma \\ &= mc^2\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}\cdots\right) \\ &\approx mc^2+\frac{1}{2}mv^2+\frac{3}{8}mv^2\frac{v^2}{c^2}\end{aligned}$$

The third term is negligible at ordinary velocities, while we recognize the second term as the usual Newtonian kinetic energy. We therefore identify $E = p^0c$. Since the first term is constant it plays no measurable role in classical mechanics but it suggests that there is intrinsic energy associated with the mass of an object. This conjecture is confirmed by observations of nuclear decay. In such decays, the mass of the initial particle is greater than the sum of the masses of the product particles, with the energy difference

$$\Delta E = m_{initial}c^2 - \sum m_{final}c^2$$

correctly showing up as kinetic energy.

Notice the difference between the mass and energy of a particle. The *mass* is independent of frame of reference – the mass of an electron is always 511 keV – but the energy, $E = mc^2\gamma$, depends on the velocity. In earlier texts the factor of γ is often associated with the mass, but this is not consistent with $p^\alpha = mu^\alpha$ for 4-vectors p^α, u^α.

3 Acceleration

Next we consider acceleration. We define the acceleration 4-vector to be the proper-time rate of change of 4-velocity,

$$\begin{aligned} a^\alpha &= \frac{du^\alpha}{d\tau} \\ &= \frac{dt}{d\tau}\frac{d\left(\gamma\left(c, v^i\right)\right)}{dt} \\ &= \gamma\left(-\frac{1}{2}\gamma^3\left(-2\frac{v^m a_m}{c^2}\right)\left(c, v^i\right) + \gamma\left(0, a^i\right)\right) \\ &= \frac{v^m a_m}{c^2}\gamma^3 u^\alpha + \gamma^2\left(0, a^i\right) \end{aligned}$$

Is this consistent with our expectations?

If we are in the instantaneous rest frame of the particle (or the particle momentarily at rest in our frame) then $v^i = 0$ and

$$a^\alpha = \left(0, a^i\right)$$

and the first term remains small except when $v^m a_m$ approaches c^2. This is consistent with classical experiments.

We also know that

$$u^\alpha u_\alpha = -c^2$$

which means that

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(-c^2\right) \\ &= 2\frac{du^\alpha}{d\tau}u_\alpha \end{aligned}$$

Therefore, the 4-velocity and 4-acceleration are orthogonal, which we easily verify directly,

$$\begin{aligned} u^\alpha a_\alpha &= u_\alpha\left(\frac{v^m a_m}{c^2}\gamma^3 u^\alpha + \gamma^2\left(0, a^i\right)\right) \\ &= \left(\frac{v^m a_m}{c^2}\gamma^3\left(-c^2\right) + \gamma\left(-c, v_i\right)\cdot\gamma^2\left(0, a^i\right)\right) \\ &= -v^m a_m\gamma^3 + \gamma^3 a^i v_i \\ &= 0 \end{aligned}$$

Now compute $a^\alpha a_\alpha$:

$$\begin{aligned} a^\alpha a_\alpha &= \left(\frac{v^m a_m}{c^2}\gamma^3 u^\alpha + \gamma^2\left(0, a^i\right)\right)a_\alpha \\ &= \gamma^2\left(0, a^i\right)a_\alpha \\ &= \gamma^2\left(0, a^i\right)\cdot\left(\frac{v^m a_m}{c^2}\gamma^4\left(c, v_i\right) + \gamma^2\left(0, a_i\right)\right) \\ &= \frac{v^m a_m}{c^2}\gamma^6 a^i v_i + \gamma^4 a^i a_i \\ &= \gamma^4\left(a^i a_i + \gamma^2\frac{\left(v^m a_m\right)^2}{c^2}\right) \end{aligned}$$

This expression gives the acceleration of a particle moving with relative velocity v^i when the acceleration in the instantaneous rest frame of the particle is given by the $v^i = 0$ expression

$$a^\alpha a_\alpha = a^i a_i$$

We consider two cases. First, suppose v^i is parallel to a^i. Then since $a^\alpha a_\alpha$ is invariant, the 3-acceleration is given by

$$\begin{aligned} a^\alpha a_\alpha &= \gamma^4 \left(a^2 + \gamma^2 \frac{v^2 a^2}{c^2} \right) \\ &= \gamma^6 a^i a_i \end{aligned}$$

or

$$a^i a_i = a^\alpha a_\alpha \left(1 - \frac{v^2}{c^2} \right)^3$$

where $a^\alpha a_\alpha$ is independent of v^i. Therefore, as the particle nears the speed of light, its apparent 3-acceleration decreases dramatically. When the acceleration is orthogonal to the velocity, the exponent is reduced,

$$a^i a_i = a^\alpha a_\alpha \left(1 - \frac{v^2}{c^2} \right)^2$$

4 Equations of motion from an action

4.1 Free particle

The relativistic action for a free particle is surprisingly simple. To derive a suitable equation of motion, we once again start with arc length. Suppose we have a timelike curve $x^\alpha(\lambda)$. Then distance along the curve is given by

$$\tau = -\frac{1}{c^2} \int \sqrt{(-v^\alpha v_\alpha)} d\lambda$$

where

$$v^\alpha = \frac{dx^\alpha}{d\lambda}$$

Since the integral is reparameterization invariant, there is no loss of generality if we use the 4-velocity in place of v^a and write

$$\tau_C = -\frac{1}{c^2} \int_C \sqrt{(-u^\alpha u_\alpha)} d\tau$$

Then the path of extremal proper time is given by the Euler-Lagrange equation

$$\frac{d}{d\tau} \frac{\partial}{\partial u^\beta} \left(-\frac{1}{c^2} u^\alpha u_\alpha \right) = 0$$

that is, vanishing 4-acceleration,

$$\frac{du^\alpha}{d\tau} = 0$$

4.2 Relativistic action with a potential

We can easily generalize this expression to include a potential. For relativistic problems it is possible to keep the action reparameterization invariant. To do so, we must multiply the line element by a function instead of adding the function. This gives

$$\tau_C = \frac{1}{c} \int_C \phi \sqrt{(-u^\alpha u_\alpha)} d\tau$$

The Euler-Lagrange equation is

$$\begin{aligned}\frac{d}{d\tau}\left(-\phi\left(-u^{\alpha}u_{\alpha}\right)^{-1/2}u_{\alpha}\right)-\left(-u^{\alpha}u_{\alpha}\right)^{1/2}\frac{\partial\phi}{\partial x^{\alpha}} &= 0\\ \frac{1}{c^{2}}\frac{d}{d\tau}\left(\phi u_{\alpha}\right)+\frac{\partial\phi}{\partial x^{\alpha}} &= 0\end{aligned}$$

where we have simplified using the normalization condition $c=\left(-u^{\alpha}u_{\alpha}\right)^{1/2}$. Expanding the derivatives, and rearranging,

$$\begin{aligned}0 &= \frac{1}{c^{2}}\frac{du_{\alpha}}{d\tau}\phi+\frac{1}{c^{2}}u_{\alpha}\frac{d\phi}{d\tau}+\frac{\partial\phi}{\partial x^{\alpha}}\\ &= \frac{1}{c^{2}}\frac{du_{\alpha}}{d\tau}\phi+\frac{1}{c^{2}}u_{\alpha}\frac{dx^{\beta}}{d\tau}\frac{\partial\phi}{\partial x^{\beta}}+\delta_{\alpha}^{\beta}\frac{\partial\phi}{\partial x^{\beta}}\\ &= \frac{1}{c^{2}}\frac{du_{\alpha}}{d\tau}\phi+\left(\frac{1}{c^{2}}u^{\beta}u_{\alpha}+\delta_{\alpha}^{\beta}\right)\frac{\partial\phi}{\partial x^{\beta}}\end{aligned}$$

Notice that

$$P^{\beta}{}_{\alpha}=\delta_{\alpha}^{\beta}+\frac{1}{c^{2}}u^{\beta}u_{\alpha}$$

is a projection operator, because

$$\begin{aligned}P^{\mu}{}_{\beta}P^{\beta}{}_{\alpha} &= \left(\delta_{\beta}^{\mu}+\frac{1}{c^{2}}u^{\mu}u_{\beta}\right)\left(\delta_{\alpha}^{\beta}+\frac{1}{c^{2}}u^{\beta}u_{\alpha}\right)\\ &= \delta_{\beta}^{\mu}\delta_{\alpha}^{\beta}+\frac{1}{c^{2}}\delta_{\beta}^{\mu}u^{\beta}u_{\alpha}+\frac{1}{c^{2}}u^{\mu}u_{\beta}\delta_{\alpha}^{\beta}+\frac{1}{c^{4}}u^{\mu}u_{\beta}u^{\beta}u_{\alpha}\\ &= \delta_{\alpha}^{\mu}+\frac{1}{c^{2}}u^{\mu}u_{\alpha}+\frac{1}{c^{2}}u^{\mu}u_{\alpha}-\frac{1}{c^{2}}u^{\mu}u_{\alpha}\\ &= P^{\mu}{}_{\alpha}\end{aligned}$$

Indeed, it projects into directions orthogonal to the 4-velocity, giving zero when we act on u^{β}:

$$\begin{aligned}P^{\alpha}{}_{\beta}u^{\beta} &= \left(\delta_{\beta}^{\alpha}+\frac{1}{c^{2}}u^{\alpha}u_{\beta}\right)u^{\beta}\\ &= \left(u^{\alpha}+\frac{1}{c^{2}}u^{\alpha}\left(u_{\beta}u^{\beta}\right)\right)\\ &= 0\end{aligned}$$

Notice that the projection is symmetric, $P^{\alpha}{}_{\beta}=P_{\beta}{}^{\alpha}$

Now we may write the equation of motion as

$$\frac{1}{c^{2}}\frac{du_{\alpha}}{d\tau}\phi=-P_{\alpha}{}^{\beta}\frac{\partial\phi}{\partial x^{\beta}}$$

The projection operator is necessary because the acceleration term is orthogonal to u^{α}. Dividing by $\frac{\phi}{c^{2}}$, we see that

$$\frac{du_{\alpha}}{d\tau}=-c^{2}P_{\alpha}{}^{\beta}\frac{\partial\ln\phi}{\partial x^{\beta}}$$

If we identify

$$\phi=\exp\left(\frac{V}{mc^{2}}\right)$$

then we arrive at the desired equation of motion

$$m\frac{du_{\alpha}}{d\tau}=-P_{\alpha}{}^{\beta}\frac{\partial V}{\partial x^{\beta}}$$

which now is seen to follow as the extremum of the functional

$$S[x^a] = \frac{1}{c}\int_C e^{\frac{V}{mc^2}} \left(-u^\alpha u_\alpha\right)^{1/2} d\tau \tag{5}$$

See the exercises for other ways of arriving at this result.

It is suggestive to notice that the integrand is simply the usual line element multiplied by a scale factor.

$$\begin{aligned} d\sigma^2 &= \frac{1}{c^2} e^{\frac{2V}{mc^2}} \left(-u^\alpha u_\alpha\right) d\tau^2 \\ &= -e^{\frac{2V}{mc^2}} ds^2 \end{aligned}$$

This is called a *conformal line element* because it is formed from a metric which is related to the flat space metric by a conformal factor, $e^{\frac{V}{mc^2}}$,

$$g_{\alpha\beta} = e^{\frac{2V}{mc^2}} \eta_{\alpha\beta}$$

Conformal transformations also appear in the study of Hamiltonian mechanics.

We can generalize the action further by observing that the potential is the integral of the force along a curve,

$$V = -\int_C F_\alpha dx^\alpha$$

The potential is defined only when this integral is single valued. By Stoke's theorem, this occurs if and only if the force is curl-free. But even for general forces we can write the action as

$$S[x^a] = \frac{1}{c}\int_C e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(-u^\alpha u_\alpha\right)^{1/2} d\tau$$

In this case, variation leads to

$$\begin{aligned} 0 &= \delta S[x^a] \\ &= \frac{1}{c}\int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta\right)\delta x^\alpha\right) \left(-u^\alpha u_\alpha\right)^{1/2} d\tau \\ &\quad + \frac{1}{c}\int_C e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \frac{1}{2\left(-u^\alpha u_\alpha\right)^{1/2}} \left(-2u_\alpha \delta u^\alpha\right) d\tau \end{aligned}$$

Now, replacing $-u^\alpha u_\alpha = c^2$, and integrating the second term by parts,

$$\begin{aligned} 0 &= \int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta\right)\delta x^\alpha\right) d\tau \\ &\quad + \int_C \frac{d}{d\tau}\left(e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \frac{1}{c^2} u_\alpha\right)\delta x^\alpha d\tau \\ &= \int_C \left(-\frac{1}{mc^2} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \left(\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta\right)\delta x^\alpha\right) d\tau \\ &\quad + \int_C \left(\frac{1}{c^2} u_\alpha \frac{d}{d\tau} e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} + e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} \frac{1}{c^2}\frac{du_\alpha}{d\tau}\right)\delta x^\alpha d\tau \end{aligned}$$

We need to evaluate the derivatives of the integrals, remembering that the integral is along the curve C with tangent u^α. For the first,

$$\frac{\partial}{\partial x^\alpha}\int_C F_\beta dx^\beta = F_\alpha$$

while for the second,

$$\begin{aligned}
\frac{d}{d\tau}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha} &= -\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{d}{d\tau}\int_C F_\alpha dx^\alpha \\
&= -\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{d}{d\tau}\int_C F_\alpha \frac{dx^\alpha}{d\tau}d\tau \\
&= -\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}F_\alpha u^\alpha
\end{aligned}$$

Combining these with the variation,

$$\begin{aligned}
0 &= \int_C \left(-\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}F_\alpha \delta x^\alpha\right)d\tau \\
&\quad+\int_C \left(\frac{1}{c^2}u_\alpha\left(-\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\mu dx^\mu}F_\beta u^\beta\right)+e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\frac{1}{c^2}\frac{du_\alpha}{d\tau}\right)\delta x^\alpha d\tau \\
&= -\frac{1}{mc^2}\int_C e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}\left(F_\alpha+\frac{1}{c^2}u_\alpha u^\beta F_\beta - m\frac{du_\alpha}{d\tau}\right)\delta x^\alpha d\tau
\end{aligned}$$

Cancelling the overall factor of $-\frac{1}{mc^2}e^{-\frac{1}{mc^2}\int_C F_\alpha dx^\alpha}$, the equation of motion is

$$m\frac{du_\alpha}{d\tau} = F_\alpha + \frac{1}{c^2}u_\alpha u^\beta F_\beta$$

and therefore,

$$m\frac{du_\alpha}{d\tau} = P_\alpha^{\ \ \beta}F_\beta$$

This time the equation holds for an arbitrary relativistic force.

Finally, consider the non-relativistic limit of the action. If $v \ll c$ and $V \ll mc^2$ then to lowest order,

$$\begin{aligned}
S[x^a] &= \int_C e^{\frac{V}{mc^2}}d\tau \\
&= \int_C \left(1+\frac{V}{mc^2}\right)\frac{1}{\gamma}dt \\
&= \frac{1}{mc^2}\int_C \left(mc^2+V\right)\sqrt{1-\frac{v^2}{c^2}}dt \\
&= -\frac{1}{mc^2}\int_C \left(mc^2\left(-1+\frac{v^2}{2c^2}\right)-V\right)dt \\
&= -\frac{1}{mc^2}\int_C \left(-mc^2+\frac{1}{2}mv^2-V\right)dt
\end{aligned}$$

Discarding the multiplier and irrelevant constant mc^2 in the integral we recover

$$S_{Cl} = \int_C \left(\frac{1}{2}mv^2 - V\right)dt = \int_C (T-V)\,dt$$

Since the conformal line element is a more fundamental object than the classical action, this may be regarded as another derivation of the classical form of the Lagrangian, $L = T - V$.

Exercises

1. Suppose a muon is produced in the upper atmosphere moving downward at $v = .99c$ relative to the surface of Earth. If it decays after a proper time $\tau = 2.2 \times 10^{-6}$ seconds, how far would it travel if there were no time dilation? Would it reach Earth's surface? How far does it actually travel relative to Earth? Note that many muons are seen reaching Earth's surface.

2. A free neutron typically decays into a proton, an electron, and an antineutrino. How much kinetic energy is shared by the final particles?

3. Suppose a proton at Fermilab travels at $.99c$. Compare Newtonian energy, $\frac{1}{2}mv^2$ to the relativistic energy p^0c.

4. A proton at the LHC at CERN may currently be given an energy of 7 TeV. What is the speed of the proton?

5. A projection operator is an operator which is idempotent, that is, it is its own square.

   (a) Write $P^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta} + \frac{1}{c^2}u^{\alpha}u_{\beta}$ as a matrix in the rest frame (i.e., the inertial frame where u^{α} is simply $u^{\alpha} = (c, \mathbf{0})$.

   (b) We showed that $P^{\beta}{}_{\alpha}u^{\alpha} = 0$. Show that if $u_{\alpha}w^{\alpha} = 0$, that $P^{\alpha}{}_{\beta}w^{\beta} = w^{\alpha}$.

6. Show that the 4-velocity takes the form $u^{\alpha} = (c, \mathbf{0})$ if and only if the 3-velocity vanishes.

7. Consider the action

$$S[x^a] = \int (mu^{\alpha}u_{\alpha} + \phi)\, d\tau$$

   This is no longer reparameterization invariant, so we need an additional Lagrange multiplier term to enforce the constraint,

$$+\lambda \left(u^{\alpha}u_{\alpha} + c^2\right)$$

   so the action becomes

$$S_1[x^a] = \int \left(mu^{\alpha}u_{\alpha} + c^2\phi + \lambda\left(u^{\alpha}u_{\alpha} + c^2\right)\right) d\tau$$

   (a) Write the Euler-Lagrange equations (including the one arising from the variation of λ).

   (b) The constraint implies $u^{\alpha}\frac{du_{\alpha}}{d\tau} = 0$. Solve for λ by contracting the equation of motion with u^{α}, using $u^{\alpha}\frac{du_{\alpha}}{d\tau} = 0$, and integrating. You should find that

$$\lambda = -\frac{1}{2}(\phi + a)$$

   (c) Substitute λ back into the equation of motion and show that the choice

$$\ln\left(\frac{\phi - 2m + a}{\phi_0 - 2m + a}\right) = \frac{1}{mc^2}V$$

   gives the correct equation of motion.

   (d) Show, using the constraint freely, that S_1 is a multiple of the action of eq.(5).

8. Consider the action

$$S_2[x^a] = \int \left(mc^2\sqrt{1 - \frac{v^2}{c^2}} + V\right) dt$$

   Show that S_2 has the correct low-velocity limit, $L = T - V$. Show that the Euler-Lagrange equation following from the variation of S_2 is *not* covariant. S_2 is therefore unsatisfactory.

9. Consider the action

$$S_3[x^a] = \int (mu^{\alpha}u_{\alpha} - 2V)\, d\tau$$

(a) Show that the Euler-Lagrange equation for S_3 is

$$\frac{d}{d\tau} m u_\alpha = -\frac{\partial V}{\partial x^\alpha}$$

(b) Show that the constraint, $u^\alpha u_\alpha = -c^2$ is not satisfied for general potentials V.

(c) Show that S_3 has the wrong low-velocity limit.