Problems in the Schwarzschild geometry

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Start work on the problems below. See the notes on Schwarzschild geodesics to get started. Help is available.

Summary of geodesics:

We have, for timelike curves:

$$\begin{aligned}
\frac{dt}{d\tau} &= u_0^0\left(\frac{1-\frac{2M}{r_0}}{1-\frac{2M}{r}}\right)\\
\frac{d\varphi}{d\tau} &= \frac{L}{r^2}\\
\frac{dr}{d\tau} &= \sqrt{2E+\frac{2M}{r}-\frac{L^2}{r^2}+\frac{2ML^2}{r^3}}
\end{aligned}$$

where

$$\begin{aligned}
L &= r_0^2\left(\frac{d\varphi}{d\tau}\right)_0\\
E &= -\frac{1}{2}\left[1-\left(1-\frac{2M}{r_0}\right)^2\left(u_0^0\right)^2\right]
\end{aligned}$$

Problems:

1. Find the form of $\frac{dr}{d\tau}$ for spacelike and null geodesics.
2. Find the proper time required for a particle to fall from radius $r_0 > 2M$ to $r = 2M$. Evaluate the time numerically for a solar mass black hole, and a galactic black hole with mass 10^7 times the mass of the sun.
3. Find the proper time required for a particle to fall from radius $r_0 = 2M$ to $r = 0$. Evaluate for solar mass and 10^7 solar mass black holes.

Find predictions for the three classical tests of general relativity. For all three problems, assume $\frac{2M}{r} \ll 1$.

1. **Perihelion advance of Mercury**. From the orbital equation derived by integrating $\frac{dr}{d\varphi}$ (given in the Notes), find the angle between adjacent minima of $r(\varphi)$. The amount by which this exceeds 2π is the perihelion advance.

2. **Gravitational red shift**. Use your expression for null geodesics, restricted to outward radial motion ($L = 0$) to find the fractional change in frequency $\frac{\omega'}{\omega}$ (or wavelength) of the light. Remember that the momentum 4-vector, $p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right) = \hbar\left(\frac{\omega}{c}, \mathbf{k}\right)$ is tangent to the null curve.

3. **Deflection of light passing the sun**. Again consider a null geodesic but now $L \neq 0$. Instead, L follows from the point of closest approach, $L = r_0 c$ where r_0 is the impact parameter. The orbit is symmetric around this point, so the integral for r will run from ∞ to r_0 and back out again. Write the solution as $\varphi = 2\int_{r_0}^{\infty} F(r)\, dr$ where $F(r)$ is what you get from the null geodesic equation. The deviation from a straight line is $\Delta\varphi = 2\int_{r_0}^{\infty} F(r)\, dr - \pi$.