Schwarzschild geodesics

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1 Schwarzschild spacetime

Spherically symmetric solutions to the Einstein equation take the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

where $G = c = 1$ and the mass $M = \frac{GM}{c^2}$ is determined from the asymptotic acceleration. We also have the connection components, which now take the form:

$$\begin{aligned}
\Gamma^0_{01} &= \Gamma^0_{10} = \frac{M}{r^2\left(1 - \frac{2M}{r}\right)} \\
\Gamma^1_{00} &= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \\
\Gamma^1_{11} &= -\frac{M}{r^2\left(1 - \frac{2M}{r}\right)} \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r} \\
\Gamma^1_{22} &= -r\left(1 - \frac{2M}{r}\right) \\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^1_{33} &= -\left(1 - \frac{2M}{r}\right) r \sin^2\theta \\
\Gamma^3_{23} &= \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta
\end{aligned}$$

Despite the *coordinate* singularity at $r = 2M$, the only true geometric singularity is at the origin, $r = 0$.

2 The geodesic equations

Now consider the geodesic equation,

$$0 = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu$$

This equation determines the paths of freely falling particles in the neighborhoods of a planet, star or other spherically symmetric source. We also have the norm of the 4-velocity, $u^\mu = \frac{dx^\alpha}{d\tau}$,

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1 - \frac{2M}{r}}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 \tag{1}$$

We may also find spacelike geodesics by setting the norm of the unit tangent vector, $t^\alpha = \frac{dx^\alpha}{dx}$ to $+1$, respectively. For null geodesics we cannot use τ for the parameter since $d\tau = 0$, but setting $t^\alpha = \frac{dx^\alpha}{d\lambda}$ for any other suitable parameter, we have $t^\alpha t_\alpha = 0$.

Substituting for the connection coefficients, the geodesic equation for each component of the 4-velocity (or tangent vector t^α) is then,

$$
\begin{aligned}
\frac{du^0}{d\tau} &= -\frac{2M}{r^2\left(1-\frac{2M}{r}\right)}u^0u^1 \\
\frac{du^1}{d\tau} &= -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)u^0u^0 + \frac{M}{r^2\left(1-\frac{2M}{r}\right)}u^1u^1 + r\left(1-\frac{2M}{r}\right)u^2u^2 + \left(1-\frac{2M}{r}\right)r\sin^2\theta u^3u^3 \\
\frac{du^2}{d\tau} &= -\frac{2}{r}u^2u^1 + \sin\theta\cos\theta u^3u^3 \\
\frac{du^3}{d\tau} &= -\frac{2}{r}u^1u^3 - \frac{2\cos\theta}{\sin\theta}u^2u^3
\end{aligned}
$$

A judicious choice of coordinates can simplify these equations. Because the spacetime is spherically symmetric, we may rotate in the θ and φ coordinates arbitrarily. Let the initial point of the geodesic be arbitrary. Then we may rotate the θ and φ coordinates so tht the initial point has $\theta = \frac{\pi}{2}$ and $\varphi = 0$. We may also choose the origin of the time coordinate arbitrarily, so the makes initial position is

$$x_0^\alpha = \left(0, r_0, \frac{\pi}{2}, 0\right)$$

Once this point is fixed there remains a single rotation about the initial point. If the initial 4-velocity has components u_0^2, u_0^3 tangent to the spheres of symmetry, we may rotate the coordinates until $u_0^2 = 0$, so the arbitrary initial velocity may be taken to be

$$u_0^\alpha = \left(u_0^0, u_0^1, 0, u_0^3\right)$$

With these conventions, the initial u^2 acceleration is

$$
\begin{aligned}
\left.\frac{du^2}{d\tau}\right|_0 &= -\frac{2}{r}u_0^2u_0^1 + \sin\frac{\pi}{2}\cos\frac{\pi}{2}\left(u_0^3\right)^2 \\
&= 0
\end{aligned}
$$

Since there is no initial acceleration or velocity, $\theta = \frac{\pi}{2}$ at all times. The u^2 equation is solved and the remaining geodesic equations reduce to

$$
\begin{aligned}
\frac{du^0}{d\tau} &= -\frac{2M}{r^2\left(1-\frac{2M}{r}\right)}u^0u^1 \\
\frac{du^1}{d\tau} &= -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)u^0u^0 + \frac{M}{r^2\left(1-\frac{2M}{r}\right)}u^1u^1 + \left(1-\frac{2M}{r}\right)ru^3u^3 \\
\frac{du^3}{d\tau} &= -\frac{2}{r}u^1u^3
\end{aligned}
$$

This simplification occurs because of conservation of angular momentum. The direction of the angular momentum vector is constant, so if the coordinates are chosen so that the motion starts in the equatorial plane, it must stay there. This is not true for a rotating black hole, which generically puts a torque on the particle.

There is another simplification we can make by considering the norm of the 4-velocity, eq.1. Differentiating with respect to τ, eq.1 becomes

$$
\begin{aligned}
0 &= -2\left(1-\frac{2M}{r}\right)u^0\frac{du^0}{d\tau} - \left(\frac{2M}{r^2}u^1\right)\left(u^0\right)^2 + \frac{2}{1-\frac{2M}{r}}u^1\frac{du^1}{d\tau} \\
&\quad -\frac{1}{\left(1-\frac{2M}{r}\right)^2}\frac{2M}{r^2}\left(u^1\right)^3 + 2r^2u^3\frac{du^3}{d\tau} + 2ru^1\left(u^3\right)^2
\end{aligned}
$$

This holds for spacelike or null geodesics as well, with the appropriate parameterizations. Substituting the expressions for $\frac{du^0}{d\tau}$ and $\frac{du^3}{d\tau}$ from the geodesic equation and cancelling common factors,

$$
\begin{aligned}
0 &= \frac{4M}{r^2}\left(u^0\right)^2 u^1 - \frac{2M}{r^2}u^1\left(u^0\right)^2 + \frac{2}{1-\frac{2M}{r}}u^1\frac{du^1}{d\tau} \\
&\quad - \frac{1}{\left(1-\frac{2M}{r}\right)^2}\frac{2M}{r^2}\left(u^1\right)^3 - 4r\left(u^3\right)^2 u^1 + 2ru^1\left(u^3\right)^2 \\
0 &= \frac{M}{r^2}\left(u^0\right)^2 + \frac{1}{1-\frac{2M}{r}}\frac{du^1}{d\tau} - \frac{1}{\left(1-\frac{2M}{r}\right)^2}\frac{M}{r^2}\left(u^1\right)^2 - r\left(u^3\right)^2
\end{aligned}
$$

so that

$$
\frac{du^1}{d\tau} = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\left(u^0\right)^2 + \frac{M}{r^2\left(1-\frac{2M}{r}\right)}\left(u^1\right)^2 + r\left(1-\frac{2M}{r}\right)\left(u^3\right)^2
$$

reproducing the radial geodesic equation. We may therefore replace the radial equation by the norm of the 4-velocity.

The final set of equations to solve is:

$$
\frac{du^0}{d\tau} = -\frac{2M}{r^2\left(1-\frac{2M}{r}\right)}u^0u^1 \tag{2}
$$

$$
\frac{du^3}{d\tau} = -\frac{2}{r}u^1u^3 \tag{3}
$$

$$
\left.\begin{array}{cc} (timelike) & -1 \\ (null) & 0 \\ (spacelike) & +1 \end{array}\right\} = -\left(1-\frac{2M}{r}\right)\left(u^0\right)^2 + \frac{1}{1-\frac{2M}{r}}\left(u^1\right)^2 + r^2\left(u^3\right)^2 \tag{4}
$$

3 Solution

Eqs.(2) and (3) are straightforward to integrate. For eq.(2), divide by $u^0$and write $u^1 = \frac{dr}{d\tau}$,

$$
\begin{aligned}
\frac{1}{u^0}\frac{du^0}{d\tau} &= -\frac{2M}{r^2\left(1-\frac{2M}{r}\right)}\frac{dr}{d\tau} \\
\int_{u^0_0}^{u^0}\frac{1}{u^0}du^0 &= -\int_{r_0}^{r}\frac{2M}{r\left(r-2M\right)}dr \\
\ln\frac{u^0}{u^0_0} &= \int_{r_0}^{r}\left(\frac{1}{r}-\frac{1}{r-2M}\right)dr \\
&= \ln\frac{r}{r_0} - \ln\frac{r-2M}{r_0-2M} \\
&= \ln\frac{r\left(r_0-2M\right)}{r_0\left(r-2M\right)}
\end{aligned}
$$

so exponentiating,

$$
u^0 = u^0_0\left(\frac{1-\frac{2M}{r_0}}{1-\frac{2M}{r}}\right)
$$

The solution for u^3 is found the same way, dividing by u^3 and writing $u^1 = \frac{dr}{d\tau}$,

$$\begin{aligned} \frac{1}{u^3}\frac{du^3}{d\tau} &= -\frac{2}{r}\frac{dr}{d\tau} \\ \ln\frac{u^3}{u_0^3} &= -\int_{r_0}^{r}\frac{2}{r}dr \\ &= -2\ln\frac{r}{r_0} \end{aligned}$$

Exponentiation yields

$$r^2u^3 = r_0^2u_0^3 \equiv L$$

This is the angular momentum per unit mass. We may replace u^3 using $u^3 = \frac{L}{r^2}$.

Finally, choosing the timelike case and using the results for u^0 and u^3, eq.4 gives u^1:

$$\begin{aligned} -1 &= -\left(1-\frac{2M}{r}\right)\left(u^0\right)^2 + \frac{1}{1-\frac{2M}{r}}\left(u^1\right)^2 + r^2\left(u^3\right)^2 \\ \frac{1}{1-\frac{2M}{r}}\left(u^1\right)^2 &= \left(1-\frac{2M}{r}\right)\left(u^0\right)^2 - 1 - r^2\left(u^3\right)^2 \\ \left(u^1\right)^2 &= \left(1-\frac{2M}{r}\right)^2\left(u^0\right)^2 - \left(1-\frac{2M}{r}\right) - r^2\left(1-\frac{2M}{r}\right)\left(u^3\right)^2 \end{aligned}$$

so that with $u^1 = \frac{dr}{d\tau}$ and substituting for u^0 and u^3,

$$\frac{dr}{d\tau} = \sqrt{\left(1-\frac{2M}{r_0}\right)^2\left(u_0^0\right)^2 - 1 + \frac{2M}{r} - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}}$$

Define the initial constant to be $2E$

$$\begin{aligned} E &\equiv -\frac{1}{2}\left[1-\left(1-\frac{2M}{r_0}\right)^2\left(u_0^0\right)^2\right] \\ &= -\frac{M}{r} + \frac{1}{2}\left(u^1\right)^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)r^2\left(u^3\right)^2 \end{aligned}$$

so that E is the (proper time) energy per unit mass. The form now simplifies to

$$\frac{dr}{d\tau} = \sqrt{2E + \frac{2M}{r} - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}}$$

and we have reduced the problem to quadratures.

Compare this to the Newtonian result (see Appendix):

$$\dot{r} = \sqrt{2E - \frac{L^2}{r^2} + \frac{2GM}{r}}$$

In addition to the expression in terms of τ rather than t, where $\frac{dt}{d\tau} = \frac{\sqrt{2E+1}}{1-\frac{2M}{r}}$, the results differ only by the final term $\frac{2ML^2}{r^3}$. This term is generally small, and we can use what we have learned about Newtonian orbits to obtain solutions.

Restoring G and c, we compare the magnitudes of the terms in the line element. We have already seen that

$$\frac{2M}{r} = \frac{2GM}{rc^2} = \frac{v_{escape}^2}{c^2}$$

where v_{escape} is the *classical* escape velocity (there is no escape velocity from within $r = 2M$!). This term is less than 1 until the particle reaches the horizon at $r = 2M$. We also estimate

$$\frac{L^2}{r^2} = \frac{L^2}{r^2 c^2} = \left(\frac{dt}{d\tau}\right)^2 \frac{r^2 \dot{\varphi}^2}{c^2} = \left(\frac{dt}{d\tau}\right)^2 \frac{v_\varphi^2}{c^2}$$

which is also less than 1 for nonrotating black holes. For the final term,

$$\begin{aligned} \frac{2GML^2}{r^3 c^2} &= \frac{2GM}{rc^2} \times \frac{L^2}{c^2 r^2} \\ &\sim \frac{v_{escape}^2}{c^2} \times \frac{v_{orbital}^2}{c^2} \end{aligned}$$

For ordinary velocities and solar system gravity, this is a small correction. Any orbits for which this is a small correction may be treated perturbatively.

Summary of geodesics:

We have, for timelike curves:

$$\begin{aligned} \frac{dt}{d\tau} &= u_0^0 \left(\frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}}\right) \\ \frac{d\varphi}{d\tau} &= \frac{L}{r^2} \\ \frac{dr}{d\tau} &= \sqrt{2E + \frac{2M}{r} - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}} \end{aligned}$$

where

$$\begin{aligned} L &= r_0^2 \left(\frac{d\varphi}{d\tau}\right)_0 \\ E &= -\frac{1}{2}\left[1 - \left(1 - \frac{2M}{r_0}\right)^2 \left(u_0^0\right)^2\right] \end{aligned}$$

Exercise: Find the form of $\frac{dr}{d\tau}$ for spacelike and null geodesics (see problems).

4 The radial integral

As with the Newtonian case (see Appendix), it is much more revealing to solve for $r(\varphi)$than for $r(\tau)$. Therefore, with

$$\begin{aligned} \frac{dr}{d\varphi} &= \frac{dr/d\tau}{d\varphi/d\tau} \\ &= \frac{r^2}{L}\dot{r} \end{aligned}$$

we integrate

$$\frac{dr}{d\varphi} = \frac{r^2}{L}\sqrt{2E - \frac{L^2}{r^2} + \frac{2GM}{r}}$$

The complete radial integral is

$$\int_0^\varphi d\varphi = \int_{r_0}^r \frac{L dr}{r^2 \sqrt{2E + \frac{2M}{r} - \frac{L^2}{r^2} + \frac{2ML^2}{r^3}}}$$

This is exactly integrable in terms of elliptic integrals, but the general form is very lengthy (try it in Wolfram!). However, except in cases of extremely strong gravity or high velocities, measurable results may be predicted perturbatively.

In cases where the motion is close to the classical solution, we factor the square root in the denominator,

$$\begin{aligned}\varphi &= \int_{r_0}^{r} \frac{Ldr}{r^2\sqrt{\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)+\frac{2ML^2}{r^3}}} \\ &= \int_{r_0}^{r} \frac{Ldr}{r^2\sqrt{2E+\frac{2M}{r}-\frac{L^2}{r^2}}\sqrt{1+\frac{2ML^2}{r^3\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)}}}\end{aligned}$$

Then, for $\frac{2ML^2}{r^3\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)} \ll 1$, we expand the second root in a Taylor series,

$$\begin{aligned}\varphi &\approx \int_{r_0}^{r} \frac{dr}{r^2\sqrt{2E+\frac{2M}{r}-\frac{L^2}{r^2}}}\left(1-\frac{ML^2}{r^3\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)}\right) \\ &= \int_{r_0}^{r} \frac{dr}{r^2\sqrt{2E+\frac{2M}{r}-\frac{L^2}{r^2}}} - \int_{r_0}^{r} \frac{ML^2dr}{r^5\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)^{3/2}}\end{aligned}$$

The first integral is the classical one,

$$\int_{r_0}^{r} \frac{dr}{\sqrt{2E+\frac{2M}{r}-\frac{L^2}{r^2}}} = \sin^{-1}\left(\frac{y}{A}\right)$$

where

$$\begin{aligned}y &= \frac{M}{L}-\frac{L}{r} \\ A^2 &= 2E+\frac{M^2}{L^2}\end{aligned}$$

For the second integral we make the same substitutions. Let $r=\frac{1}{u}$,

$$\int_{r_0}^{r} \frac{ML^2dr}{r^5\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)^{3/2}} = -\int_{r_0}^{r} \frac{ML^2u^3du}{\left(2E+2Mu-L^2u^2\right)^{3/2}}$$

Check:

$$\begin{aligned}\int_{r_0}^{r} \frac{Ldr}{r^2\sqrt{\left(2E+\frac{2M}{r}-\frac{L^2}{r^2}\right)+\frac{2ML^2}{r^3}}} &= -\int_{r_0}^{r} \frac{Ldu}{\sqrt{(2E+2Mu-L^2u^2)+2ML^2u^3}} \\ &= -\int_{r_0}^{r} \frac{Ldu}{\sqrt{(2E+2Mu-L^2u^2)}\sqrt{1+\frac{2ML^2u^3}{2E+2Mu-L^2u^2}}} \\ &= -\int_{r_0}^{r} \frac{Ldu}{\sqrt{(2E+2Mu-L^2u^2)}} + \int \frac{ML^3u^3du}{\left(2E+2Mu-L^2u^2\right)^{3/2}}\end{aligned}$$

Now write

$$\begin{aligned} \left(2E + 2Mu - L^2u^2\right) &= 2E + \frac{M^2}{L^2} - \left(Lu - \frac{M}{L}\right)^2 \\ &= A^2 - y^2 \\ y &= \frac{M}{L} - Lu \\ u &= \frac{1}{L}\left(\frac{M}{L} - y\right) \end{aligned}$$

so that

$$\varphi = \int_{r_0}^{r} \frac{dy}{\sqrt{A^2 - y^2}} - \frac{M}{L}\int \frac{\left(\frac{M}{L} - y\right)^3 dy}{\left(A^2 - y^2\right)^{3/2}}$$

Now let $y = A\sin\xi$

$$\begin{aligned} \varphi &= \int_{r_0}^{r} d\xi - \frac{M}{L}\int \frac{\left(\left(\frac{M}{L}\right)^3 - 3\left(\frac{M}{L}\right)^2 A\sin\xi + 3\frac{M}{L}A^2\sin^2\xi - A^3\sin^3\xi\right)A\cos\xi d\xi}{A^3\cos^3\xi} \\ &= \sin^{-1}\frac{y}{A} - \frac{MA}{L}\int \frac{\left(\left(\frac{M}{L}\right)^3 - 3\left(\frac{M}{L}\right)^2 A\sin\xi + 3\frac{M}{L}A^2\sin^2\xi - A^3\sin^3\xi\right)d\xi}{A^3\cos^2\xi} \\ &= \sin^{-1}\frac{y}{A} - \frac{MA}{L}\int \frac{\left(\left(\frac{M}{LA}\right)^3 + 3\frac{M}{LA}\left(1 - \cos^2\xi\right)\right)d\xi}{\cos^2\xi} + \frac{MA}{L}\int \frac{\left(3\left(\frac{M}{LA}\right)^2 + 1 - \cos^2\xi\right)\sin\xi d\xi}{\cos^2\xi} \\ &= \sin^{-1}\frac{y}{A} - \frac{MA}{L}\left(\left(\frac{M}{LA}\right)^3 + \frac{3M}{LA}\right)\int \frac{d\xi}{\cos^2\xi} + \frac{3M}{LA}\frac{MA}{L}\int \frac{\cos^2\xi d\xi}{\cos^2\xi} \\ &\quad + \frac{MA}{L}\left(3\left(\frac{M}{LA}\right)^2 + 1\right)\int \frac{\sin\xi d\xi}{\cos^2\xi} - \frac{MA}{L}\int \frac{\cos^2\xi\sin\xi d\xi}{\cos^2\xi} \\ &= \sin^{-1}\frac{y}{A} - \left(\frac{MA}{L}\right)^2\left(3 + \left(\frac{M}{LA}\right)^2\right)\tan\xi + \frac{3M^2}{L^2}\xi \\ &\quad + \frac{MA}{L}\left(1 + 3\left(\frac{M}{LA}\right)^2\right)\frac{1}{\cos\xi} + \frac{MA}{L}\cos\xi \end{aligned}$$

Substituting back,

$$\begin{aligned} \varphi &= \sin^{-1}\frac{y}{A} \\ &\quad + \frac{1}{\cos\xi}\left[-\left(\frac{MA}{L}\right)^2\left(3 + \left(\frac{M}{LA}\right)^2\right)\sin\xi + \frac{3M^2}{L^2}\xi\cos\xi + \frac{MA}{L}\left(1 + 3\left(\frac{M}{LA}\right)^2\right) + \frac{MA}{L}\cos^2\xi\right] \\ &= \left(1 + \frac{3M^2}{L^2}\right)\sin^{-1}\frac{y}{A} \\ &\quad + \frac{1}{\sqrt{1 - \frac{y^2}{A^2}}}\frac{MA}{L}\left[2 + 3\left(\frac{M}{LA}\right)^2 - \frac{MA}{L}\left(3 + \left(\frac{M}{LA}\right)^2\right)\frac{y}{A} - \frac{y^2}{A^2}\right] \end{aligned}$$

Check:

$$\varphi = \int_{r_0}^{r} \frac{Ldr}{r^2\sqrt{2E + \frac{2M}{r} - \frac{L^2}{r^2}}} - \int_{r_0}^{r} \frac{ML^3dr}{r^5\left(2E + \frac{2M}{r} - \frac{L^2}{r^2}\right)^{3/2}}$$

$$
\begin{aligned}
&= \sin^{-1}\frac{y_0}{A} - \sin^{-1}\frac{y}{A} + \frac{1}{\sqrt{1-\sin^2\xi}}\left[-M\left(1+\frac{3M^2}{L^2A^2}\right) + M\left(\frac{3M}{LA}+\frac{M^3}{L^3A^3}\right)\sin\xi + M\left(1+\frac{3M}{LA}\right)\left(1-\sin^2\xi\right)\right] \\
&= \sin^{-1}\frac{y_0}{A} - \sin^{-1}\frac{y}{A} + \frac{1}{\sqrt{1-\frac{y^2}{A^2}}}\left[-M\left(1+\frac{3M^2}{L^2A^2}\right) + M\left(\frac{3M}{LA}+\frac{M^3}{L^3A^3}\right)\frac{y}{A} + M\left(1+\frac{3M}{LA}\right)\left(1-\frac{y^2}{A^2}\right)\right] \\
&= \sin^{-1}\frac{y_0}{A} - \sin^{-1}\frac{y}{A} + \frac{3M^2}{LA\sqrt{1-\frac{y^2}{A^2}}}\left[1-\frac{M}{LA} + \left(1+\frac{3M^2}{L^2A^2}\right)\frac{y}{A} - \frac{LA}{3M}\left(1+\frac{3M}{LA}\right)\frac{y^2}{A^2}\right]
\end{aligned}
$$

Before putting in the initial conditions, we discard some small terms. The relevant constants are

$$
\begin{aligned}
\frac{M}{L} &= \frac{GM}{c^2\frac{r^2\dot\varphi}{c}} = \frac{GM}{cr^2\dot\varphi} \\
&= \frac{v_{escape}}{v_\varphi}\frac{v_{escape}}{c} \\
&\equiv \alpha v \ll 1 \\
A &= \sqrt{2E+\frac{M^2}{L^2}} \\
&\approx \sqrt{-\frac{2GM}{r} + \left(\frac{v_{escape}}{c}\frac{v_{escape}}{v_\varphi}\right)^2} \\
&\sim \frac{v_{escape}}{c}\sqrt{\left(\frac{v_{escape}}{v_\varphi}\right)^2 - 1} \\
&= v\sqrt{\alpha^2-1} \\
y &= \frac{M}{L} - \frac{L}{r} \\
&= \frac{v_{escape}}{c}\frac{v_{escape}}{v_\varphi} - \frac{v_\varphi}{c} \\
&= \frac{v_{escape}}{c}\left(\frac{v_{escape}}{v_\varphi} - \frac{v_\varphi}{v_{escape}}\right) \\
&= v\left(\alpha - \frac{1}{\alpha}\right)
\end{aligned}
$$

where we have set $v = \frac{v_{escape}}{c} \ll 1$. Let $\frac{M}{L} = \alpha\frac{v_{escape}}{c} = \alpha v$, $A = \beta v$ and $y = \left(\alpha - \frac{1}{\alpha}\right)v$. Then putting all constants into the final term,

$$
\begin{aligned}
\varphi &= \sin^{-1}\frac{y_0}{A} - \sin^{-1}\frac{y}{A} + \frac{3M^2}{LA\sqrt{1-\frac{y^2}{A^2}}}\left[1-\frac{M}{LA} + \left(1+\frac{3M^2}{L^2A^2}\right)\frac{y}{A} - \frac{LA}{3M}\left(1+\frac{3M}{LA}\right)\frac{y^2}{A^2}\right] \\
&= \sin^{-1}\frac{y_0}{A} - \sin^{-1}\frac{y}{A} + \frac{1}{\sqrt{1-\frac{y^2}{A^2}}}\alpha\beta v^2\left[2+3\left(\frac{\alpha}{\beta}\right)^2 - \alpha v\left(3+\left(\frac{\alpha}{\beta}\right)^2\right)y - \frac{y^2}{\beta^2v^2}\right] \\
&= \sin^{-1}\frac{y_0}{A} - \sin^{-1}\frac{y}{A} + \frac{1}{\sqrt{1-\frac{y^2}{\beta^2v^2}}}\left[2\alpha\beta v^2 + 3\alpha\beta v^2\left(\frac{\alpha}{\beta}\right)^2 - \alpha\beta v^2\alpha v\left(3+\left(\frac{\alpha}{\beta}\right)^2\right)y - \frac{\alpha y^2}{\beta}\right]
\end{aligned}
$$

so if we drop terms v^2 or smaller, this reduces to

$$
\frac{1}{\sqrt{1-\frac{y^2}{\beta^2v^2}}}\left[-\frac{\alpha}{\beta}y^2\right] = \frac{1}{\sqrt{1-\frac{\left(\alpha-\frac{1}{\alpha}\right)^2}{\alpha^2-1}}}\left[-\frac{\alpha}{\beta}\left(\alpha-\frac{1}{\alpha}\right)^2 v^2\right]
$$

$$= -\left(\alpha^2 - 1\right)^{3/2} v^2$$

so this is also of order $\frac{v^2_{escape}}{c^2}$ and may be dropped. Therefore,

$$\begin{aligned}
\varphi &= \left(1 + \frac{3M^2}{L^2}\right) \sin^{-1} \frac{y}{A} \\
\sin\left(\frac{L^2\varphi}{L^2 + 3M^2}\right) &= \frac{1}{\sqrt{2EL^2 + M^2}} M - \frac{1}{\sqrt{2EL^2 + M^2}} \frac{L^2}{r} \\
\frac{L^2}{r} &= \frac{\sqrt{2EL^2 + M^2}}{\sqrt{2EL^2 + M^2}} M - \sqrt{2EL^2 + M^2} \sin\left(\frac{L^2\varphi}{L^2 + 3M^2}\right) \\
r &= \frac{L^2}{M} \frac{1}{1 - \sqrt{1 + \frac{2EL^2}{M^2}} \sin\left(\frac{L^2\varphi}{L^2+3M^2}\right)}
\end{aligned}$$

Appendix: Keplerian orbits

Reduction to quadratures

For comparison, we first compute the orbits in Newtonian gravity. We start from the conservation laws. Since the velocity is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$$

the angular momentum is

$$\begin{aligned}
\overrightarrow{\mathrm{L}} &= \overrightarrow{r} \times \overrightarrow{p} \\
&= m\overrightarrow{r} \times \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}\right) \\
&= mr\dot{\theta}\overrightarrow{r} \times \hat{\theta} + mr\sin\theta\dot{\varphi}\overrightarrow{r} \times \hat{\varphi}
\end{aligned}$$

Since this is conserved in both magnitude and direction, the orbit remains in the plane perpendicular to $\overrightarrow{\mathrm{L}}$. Without loss of generality, we may take the orbit to lie in the $\theta = \frac{\pi}{2}$ plane, so that

$$\begin{aligned}
\overrightarrow{\mathrm{L}} &= mr^2\dot{\varphi}\hat{k} \\
\vec{v} &= \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}
\end{aligned}$$

The energy is also conserved,

$$\mathrm{E} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{GMm}{r}$$

Define the angular momentum per unit mass, $L = \frac{\mathrm{L}}{m} = r^2\dot{\varphi}$ and the energy per unit mass, $E = \frac{\mathrm{E}}{m}$. Then we have $\dot{\varphi} = \frac{L}{r^2}$ so that

$$\begin{aligned}
E &= \frac{1}{2}\left(\dot{r}^2 + \frac{L^2}{r^2}\right) - \frac{GM}{r} \\
\dot{r} &= -\sqrt{2E - \frac{L^2}{r^2} + \frac{2GM}{r}}
\end{aligned}$$

The negative sign is because as time increases, r decreases.

To find $r(t)$, we must integrate

$$\int_0^t dt = -\int_{r_0}^r \frac{dr}{\sqrt{2E - \frac{L^2}{r^2} + \frac{2GM}{r}}}$$

This is not difficult when there is no angular momentum, $L = 0$. The time for straight infall from rest at r_0 is, setting $L = 0$,

$$\begin{aligned} t &= -\int_{r_0}^{r} \frac{dr}{\sqrt{2E + \frac{2GM}{r}}} \\ &= -\frac{1}{\sqrt{2GM}} \int_{r_0}^{r} \frac{\sqrt{r}dr}{\sqrt{1 + \frac{Er}{GM}}} \end{aligned}$$

With these initial conditions, $E = -\frac{GM}{r_0}$, so this becomes

$$t = -\frac{1}{\sqrt{2GM}} \int_{r_0}^{r} \frac{\sqrt{r}dr}{\sqrt{1 - \frac{r}{r_0}}}$$

If we set $\frac{r}{r_0} = \sin^2 \xi < 1$ then

$$\begin{aligned} t &= -\frac{1}{\sqrt{2GM}} \int_{r_0}^{r} \frac{\sqrt{r_0 \sin^2 \xi} 2r_0 \sin \xi \cos \xi d\xi}{\sqrt{1 - \sin^2 \xi}} \\ &= -\frac{2r_0\sqrt{r_0}}{\sqrt{2GM}} \int_{r_0}^{r} \sin^2 \xi d\xi \\ &= -\frac{r_0^{3/2}}{\sqrt{2GM}} \left(\xi - \frac{1}{2} \sin 2\xi \right) \Bigg|_{r_0}^{r} \\ &= -\frac{r_0^{3/2}}{\sqrt{2GM}} \left(\sin^{-1} \sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}} \sqrt{1 - \left(\frac{r}{r_0} \right)^2} \right) \Bigg|_{r_0}^{r} \\ &= \frac{r_0^{3/2}}{\sqrt{2GM}} \left(\frac{\pi}{2} + \sqrt{\frac{r}{r_0}} \sqrt{1 - \left(\frac{r}{r_0} \right)^2} - \sin^{-1} \sqrt{\frac{r}{r_0}} \right) \end{aligned}$$

For the Newtonian case, we may take the final position to be $r = 0$, and the time for the total fall is

$$t = \frac{\pi}{2} \frac{r_0^{3/2}}{\sqrt{2GM}}$$

in agreement with Kepler's third law.

The general case is also integrable, but the result (given below) is not very enlightening. The answer is much simpler if we change from $\frac{dr}{dt}$ to $\frac{dr}{d\varphi}$, and this gives us the shape of the orbit.

Orbit shape

To find an equation for the orbit, $r(\varphi)$, divide by $\dot{\varphi} = \frac{L}{r^2}$ and integrate:

$$\varphi = \int \frac{ldr}{r^2 \sqrt{2E - \frac{L^2}{r^2} + \frac{2GM}{r}}}$$

Now set $u = \frac{1}{r}$ so that

$$\begin{aligned}\varphi &= \int \frac{-Ldu}{\sqrt{2E - L^2u^2 + 2GMu}} \\ &= \int \frac{-Ldu}{\sqrt{-\left(\frac{GM}{L} - Lu\right)^2 + 2E + \frac{G^2M^2}{L^2}}}\end{aligned}$$

Let

$$\begin{aligned}y &= \frac{GM}{L} - Lu \\ A^2 &= 2E + \frac{G^2M^2}{L^2}\end{aligned}$$

so that

$$\varphi = \int \frac{dy}{\sqrt{-y^2 + A^2}}$$

Then with $y = A\sin\theta$ we have

$$\begin{aligned}\varphi &= \int \frac{A\cos\theta d\theta}{A\cos\theta} \\ &= \theta \\ &= \arcsin\left(\frac{y}{A}\right)\end{aligned}$$

Solving for r we have,

$$\begin{aligned}A\sin\varphi &= y \\ &= \frac{GM}{L} - Lu \\ &= \frac{GM}{L} - \frac{L}{r}\end{aligned}$$

so

$$\begin{aligned}\frac{1}{r} &= \frac{GM}{L^2} - \frac{A}{L}\sin\varphi \\ r &= \frac{\frac{L^2}{GM}}{1 - \sqrt{1 + \frac{2L^2E}{G^2M^2}}\sin\varphi}\end{aligned}$$

To see that this describes an ellipse, define

$$\begin{aligned}a &\equiv \frac{L^2}{GM} \\ e &\equiv \sqrt{1 + \frac{2L^2E}{G^2M^2}}\end{aligned}$$

so that $r = \frac{a}{1-e\sin\varphi}$. Then, changing to Cartesian coordinates,

$$\begin{aligned}r - er\sin\varphi &= a \\ r - ey &= a \\ r^2 &= a^2 + 2eay + e^2y^2 \\ x^2 + y^2 &= a^2 + 2eay + e^2y^2 \\ x^2 + y^2 - 2eay - ey^2 &= a^2 \\ x^2 + (1-e)\left(y - \frac{ea}{1-e}\right)^2 &= a^2\left(1 + \frac{e^2}{1-e}\right)\end{aligned}$$

Finally, setting $y_0 = \frac{ea}{1-e}$, $b = a^2\left(1+\frac{e^2}{1-e}\right)$ and $c^2 = \frac{b^2}{1-e}$ we have the standard form for an ellipse centered at $(x, y) = (0, y_0)$:

$$\frac{x^2}{b^2} + \frac{(y-y_0)^2}{c^2} = 1$$

An examination of the magnitudes of the constants shows that this solution is valid for bound states, with $E < 0$. For positive energy, the final integral gives a hyperbolic function and the equation describes a hyperbola.

Time dependence

To find the time dependence, we have $\frac{d\varphi}{dt} = \frac{L}{r^2}$ so that

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{L}{a^2}(1-e\sin\varphi)^2 \\ \frac{Lt}{a^2} &= \int_0^{\varphi} \frac{d\varphi}{(1-e\sin\varphi)^2} \end{aligned}$$

Wolfram integrator integrates this, but does not know that $e^2 < 1$, so we must pull out factors of i everywhere, replacing $\sqrt{e^2-1} = i\sqrt{1-e^2}$. We are left with

$$\int_0^{\varphi} \frac{d\varphi}{(1-e\sin\varphi)^2} = \frac{2i}{(1-e^2)^{3/2}}\left(\tanh^{-1}\left(\frac{i\left(e-\tan\left(\frac{\varphi}{2}\right)\right)}{\sqrt{1-e^2}}\right) - \tanh^{-1}\frac{ie}{\sqrt{1-e^2}}\right) + \frac{e}{1-e^2}\left(1-\frac{\cos\varphi}{1-e\sin\varphi}\right)$$

The presence of factors of i means we need to convert from $\tanh^{-1}$ to $\tan^{-1}$ to make the solution real. Now we need

$$\begin{aligned} ix &= -\tanh iy \\ -ix &= \frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}} \\ &= i\tan y \\ x &= -\tan y \end{aligned}$$

that is,

$$i\tanh^{-1} ix = -\tan^{-1} x$$

Therefore,

$$\begin{aligned} \frac{Lt}{a^2} &= \frac{1}{(1-e^2)^{3/2}}\left(\tan^{-1}\frac{2\frac{e}{\sqrt{1-e^2}}}{1-\left(\frac{e}{\sqrt{1-e^2}}\right)^2} - \tan^{-1}\frac{2\frac{\left(e-\tan\left(\frac{\varphi}{2}\right)\right)}{\sqrt{1-e^2}}}{1-\frac{\left(e-\tan\left(\frac{\varphi}{2}\right)\right)^2}{1-e^2}}\right) + \frac{e}{1-e^2}\left(1-\frac{\cos\varphi}{1-e\sin\varphi}\right) \\ &= \frac{1}{(1-e^2)^{3/2}}\left(\tan^{-1}\frac{2e\sqrt{1-e^2}}{1-2e^2} - \tan^{-1}\frac{2\sqrt{1-e^2}\left(e-\tan\left(\frac{\varphi}{2}\right)\right)}{1-e^2-\left(e-\tan\left(\frac{\varphi}{2}\right)\right)^2}\right) + \frac{e}{1-e^2}\left(1-\frac{\cos\varphi}{1-e\sin\varphi}\right) \end{aligned}$$

Even if we could invert to find $\varphi(t)$ in closed form, the expression would likely be unwieldy.

A much simpler guide to the rate of progression is given by the conservation of angular momentum,

$$L = r^2\dot{\varphi}$$

We can interpret this by noting that in time dt the orbiting particle passes through an infinitesimal angle $d\varphi$ given by

$$\frac{L}{r^2}dt = d\varphi$$

It therefore moves through an arc length $rd\varphi$, and since this is the infinitesimal base of an isoceles triangle with sides $r, r, rd\varphi$ the infinitesimal area swept out is

$$dA = \frac{1}{2}r \cdot rd\varphi$$

Therefore, the area swept out per unit time is constant

$$\frac{dA}{dt} = \frac{1}{2}r^2 d\varphi = \frac{L}{2}$$

This is Kepler's second law.