The constant in Schwarzschild spacetime

April 1, 2015

Spherically symmetric solutions to the Einstein equation take the form

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \frac{dr^2}{1+\frac{a}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

where a is constant and $c = 1$. There is one remaining constant in this solution, and since we expect the solution to describe gravity near a spherical body, we should be able to determine its value by comparing this result with Newton's law of gravity. The comparison requires us to consider the motion of a particle in a region of small curvature and low velocity. It is sufficient to consider the initial acceleration of a particle initially at rest far from the source. This must look like the Newtonian acceleration at the same distance.

The motion of a particle is given by the geodesic equation,

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\ \mu\nu} u^\mu u^\nu$$

where u^α is the 4-velocity of the particle. Putting in the nonvanishing connection coefficients for the Schwarzschild solution, these four equations are

$$\begin{aligned}
\frac{du^0}{d\tau} &= -2\Gamma^0_{\ 01} u^0 u^1 \\
\frac{du^1}{d\tau} &= -\Gamma^1_{\ 00} u^0 u^0 - \Gamma^1_{\ 11} u^1 u^1 - \Gamma^1_{\ 22} u^2 u^2 - \Gamma^1_{\ 33} u^3 u^3 \\
\frac{du^2}{d\tau} &= -2\Gamma^2_{\ 21} u^2 u^1 - \Gamma^2_{\ 33} u^3 u^3 \\
\frac{du^3}{d\tau} &= -2\Gamma^3_{\ 13} u^1 u^3 - \Gamma^3_{\ 23} u^2 u^3
\end{aligned}$$

In order to fit the constant a, we need only the initial acceleration for a particle starting from rest. For a particle at rest, the 4-velocity is

$$\begin{aligned}
u^\alpha_0 &= \left(c\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) \\
&= \frac{dt}{d\tau}\left(c, \dot{r}, \dot{\theta}, \dot{\varphi}\right) \\
&= \frac{dt}{d\tau}(c, 0, 0, 0)
\end{aligned}$$

where the dot denotes a t derivative, $\frac{d}{dt}$. Substituting these initial values into the geodesic equations, the initial accelerations all but the radial direction vanish:

$$\left.\frac{du^0}{d\tau}\right|_{t=0} = -2\Gamma^0_{\ 01} u^0_0 u^1_0 = 0$$

$$\left.\frac{du^1}{d\tau}\right|_{t=0} = -\Gamma^1_{00}u^0_0u^0_0 - \Gamma^1_{11}u^1_0u^1_0 - \Gamma^1_{22}u^2_0u^2_0 - \Gamma^1_{33}u^3_0u^3_0 = -\left(\frac{dt}{d\tau}\right)^2_0 c^2\Gamma^1_{00}$$

$$\left.\frac{du^2}{d\tau}\right|_{t=0} = -2\Gamma^2_{21}u^2_0u^1_0 + \Gamma^2_{33}u^3_0u^3_0 = 0$$

$$\left.\frac{du^3}{d\tau}\right|_{t=0} = -2\Gamma^3_{13}u^1_0u^3_0 + \Gamma^3_{23}u^2_0u^3_0 = 0$$

Therefore, the initial acceleration is in the radial direction. We have $\frac{dt}{d\tau}$ from the norm of the 4-velocity,

$$\begin{aligned}
-1 &= g_{\mu\nu}u^\mu_0u^\nu_0 \\
&= -\left(1+\frac{a}{r}\right)\left(\frac{dt}{d\tau}_0\right)^2 + \frac{1}{1+\frac{a}{r}}\left(\frac{dr}{d\tau}_0\right)^2 + r^2\left(\frac{d\theta}{d\tau}_0\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}_0\right)^2 \\
&= -\left(1+\frac{a}{r}\right)\left(\frac{dt}{d\tau}_0\right)^2
\end{aligned}$$

so that $u^0_0 = \frac{1}{\sqrt{1+\frac{a}{r_0}}}$. Substituting this, and Γ^1_{00} from our solution

$$\begin{aligned}
\left.\frac{du^1}{d\tau}\right|_0 &= -\Gamma^1_{00}u^0_0u^0_0 \\
&= \left(\frac{1}{\sqrt{1+\frac{a}{r_0}}}\right)^2_0 c^2\frac{a}{2r_0^2}\left(1+\frac{a}{r_0}\right) \\
&= \frac{ac^2}{2r_0^2}
\end{aligned}$$

Finally, for sufficiently large initial radius,$r_0 \gg a$, we may take $\frac{dt}{d\tau} = u^0_0 = \frac{1}{\sqrt{1+\frac{a}{r_0}}} \approx 1$. This lets us replace $d\tau \approx dt$ so that $\frac{du^1}{d\tau} \approx \frac{d^2r}{dt^2}$. Comparing to the Newtonian gravitational acceleration, $\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$ we set

$$\begin{aligned}
\frac{ac^2}{2r_0^2} &= -\frac{GM}{r_0^2} \\
a &= -\frac{2GM}{c^2}
\end{aligned}$$

For a Newtonian potential, the escape velocity is given by

$$\begin{aligned}
E = 0 &= \frac{1}{2}mv^2_{escape} - \frac{GMm}{r} \\
v^2_{escape} &= \frac{2GM}{r}
\end{aligned}$$

so the factor $1 - \frac{2GM}{rc^2}$ in the line element is just $1 - \frac{v^2_{escape}}{c^2}$ where v_{escape} is the Newtonian escape velocity. For solar system masses v_{escape} is much less than the speed of light.

This completes the specification of the Schwarzschild line element,

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)dt^2 + \frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

Notice the importance of the cancellation of the mass of the particle in Newton's law:

$$\begin{aligned}
m\frac{d^2r}{dt^2} &= -\frac{GMm}{r^2} \\
\frac{d^2r}{dt^2} &= -\frac{GM}{r^2}
\end{aligned}$$

Without this, the acceleration would depend on the mass of the particle, and there would not be a single geometry that would account for all orbits. The line element is typically written choosing gravitational units,

$$\begin{aligned} G &= 1 \\ c &= 1 \end{aligned}$$

so we have the more compact expression

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

This describes gravity for objects as diverse as the moon, neutron stars, and black holes.