Riemannian Curvature

February 25, 2015

We now generalize our computation of curvature to arbitrary spaces. Before computing the form of the curvature, we note two differences from the 2-dimensional Gaussian case that we must address in higher dimensions.

1 Infinitesimal rotation transformation per unit area

First, in higher dimensions the angular change in a parallel transported vector per unit area at a point will depend on the direction of the initial vector we choose. Since the change in the vector under parallel transport is proportional to the original vector, the change in the vector after loop transport, dw^{α}, will be linear in the original vector,

$$dw^{\alpha} = w^{\beta} T^{\alpha}_{\ \beta}$$

Furthermore, since parallel transport preserves the length, $w^{\alpha} w_{\alpha}$, of vectors, $T^{\alpha}_{\ \beta}$ must satisfy

$$\begin{aligned} g_{\alpha\beta} \tilde{w}^{\alpha} \tilde{w}^{\beta} &= g_{\alpha\beta} w^{\alpha} w^{\beta} \\ g_{\alpha\beta} \left(w^{\alpha} + dw^{\alpha} \right) \left(w^{\beta} + dw^{\beta} \right) &= g_{\alpha\beta} w^{\alpha} w^{\beta} \end{aligned}$$

so to first order,

$$\begin{aligned} 0 &= g_{\alpha\beta} dw^{\alpha} w^{\beta} + g_{\alpha\beta} w^{\alpha} dw^{\beta} \\ &= g_{\alpha\beta} \left(w^{\mu} T^{\alpha}_{\ \mu} \right) w^{\beta} + g_{\alpha\beta} w^{\alpha} \left(w^{\mu} T^{\beta}_{\ \mu} \right) \\ &= g_{\alpha\beta} T^{\alpha}_{\ \mu} w^{\mu} w^{\beta} + g_{\alpha\beta} T^{\beta}_{\ \mu} w^{\alpha} w^{\mu} \\ &= 2 T_{\beta\mu} w^{\mu} w^{\beta} \end{aligned}$$

Since this must hold for all w^{α}, the symmetric part of $T_{\alpha\beta}$ must vanish and we have

$$T_{\alpha\beta} = -T_{\beta\alpha}$$

Because we transport about in infinitesimal loop, the transformation itself will be infinitesimal, $dT^{\alpha}_{\ \beta}$.

2 Area elements in higher dimensions

This infinitesimal linear transformation will also be proportional to the area of the loop about which w^{α} is transported, so we need to characterize higher dimensional areas. In three dimensions, the area of the parallelogram defined by two vectors is given by the cross product,

$$A = |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \, |\mathbf{v}| \sin\theta$$

and it is assigned the unique direction orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Writing out the components,

$$\mathbf{u} \times \mathbf{v} = \left(u^{y} v^{z} - u^{z} v^{y} \right) \mathbf{i} + \left(u^{z} v^{x} - u^{x} v^{z} \right) \mathbf{j} + \left(u^{x} v^{y} - u^{y} v^{x} \right) \mathbf{k}$$

we see that the components are just the antisymmetrized outer products

$$\frac{1}{2}\left(u^i v^j - u^j v^i\right)$$

where the factor of $\frac{1}{2}$ lets us sum freely over both indices, giving each component twice.

In higher dimensions, there is no unique direction orthogonal to two vectors, so we define an area element using this last expression,

$$A^{\alpha\beta} = \frac{1}{2}\left(u^\alpha v^\beta - u^\beta v^\alpha\right)$$

Thus, instead of identifying an area with the normal direction, we characterize it by the two directions determining its plane. A^{12} is an area in the xy plane, and so on. In curved spacetime, we will consider infinitesimal areas,

$$dA^{\alpha\beta} = \frac{1}{2}\sigma\lambda\left(u^\alpha v^\beta - u^\beta v^\alpha\right)$$

where σ, λ are infinitesimal.

The curvature is the linear transformation per unit area at a point, acting on any vector infinitesimally parallel transported about a closed loop,

$$R^\alpha_{\ \beta\mu\nu}(\mathcal{P}) \equiv \left.\frac{dT^\alpha_{\ \beta}}{dA^{\mu\nu}}\right|_{\mathcal{P}}$$

3 Parallel transport around a small closed loop

begin in a spacetime, $\left(\mathcal{M}^4, g\right)$, and compute the change in a vector, w^α, which we parallel transport around a closed loop. The loop lies in an infinitesimal 2-dimensional subspace spanned locally by a pair of vectors u^α, v^α, with pair of coordinates (λ, σ). Let the transport be along geodesics with u^α and v^α as tangents, so that

$$\begin{aligned} 0 &= u^\alpha D_\alpha u^\beta \\ \frac{du^\beta}{d\lambda} &= -u^\nu u^\mu \Gamma^\beta_{\ \mu\alpha} \end{aligned}$$

and

$$\begin{aligned} 0 &= v^\alpha D_\alpha v^\beta \\ \frac{dv^\beta}{d\sigma} &= -v^\nu v^\mu \Gamma^\beta_{\ \mu\alpha} \end{aligned}$$

The loop starts at a point $\mathcal{P}$, which we take to have coordinates $(0,0)$, progresses along a direction u^α a distance λ to a point at $(\lambda, 0)$. Next, we transport along the direction v^α a distance σ to a point at (λ, σ), giving a curve C_1 from $(0,0)$ to (λ, σ) via $(\lambda, 0)$. Rather than returning in the direction $-u^\alpha$ by λ then $-v^\alpha$ by σ, we perform a second transport from $\mathcal{P}$, interchanging the order: first v^α by σ, then u^α by λ, giving a second curve C_2 from $(0,0)$ to (λ, σ) through $(0, \sigma)$. We then compare the expressions for $w^\alpha(\lambda, \sigma)$ along the two curves. This gives the same result as if we had transported around the closed loop $C_1 - C_2$, but the computation is easier this way. The area element of the infinitesimal loop is then $dA^{\alpha\beta} = \frac{1}{2}\sigma\lambda\left(u^\alpha v^\beta - u^\beta v^\alpha\right)$.

Start with two directions, $u^\alpha(0,0)$ and $v^\alpha(0,0)$. Parallel transport each in the direction of the other to get $u^\alpha(0,\sigma)$ and $v^\alpha(\lambda, 0)$. For infinitesimal λ:

$$\begin{aligned} 0 &= u^\alpha D_\alpha v^\beta \\ &= u^\alpha\left(\frac{\partial v^\beta}{\partial x^\alpha} + v^\mu \Gamma^\beta_{\ \mu\alpha}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{dv^\beta}{d\lambda} + u^\alpha v^\mu \Gamma^\beta_{\ \mu\alpha} \\
\frac{dv^\beta}{d\lambda} &= -u^\alpha v^\mu \Gamma^\beta_{\ \mu\alpha}
\end{aligned}
$$

where all u^α, v^μ and $\Gamma^\beta_{\ \mu\alpha}$ are evaluated at $\mathcal{P}$. We also need the value of the connection at $(\lambda, 0)$:

$$
\Gamma^\beta_{\ \mu\alpha}(\lambda,0) = \Gamma^\beta_{\ \mu\alpha} + \left(u^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\lambda
$$

Now consider an arbitrary vector, w^α, at $\mathcal{P}$. Parallel transport of w^α along u^α to $(\lambda, 0)$ results in

$$
w^\beta(\lambda,0) = w^\beta - u^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\lambda
$$

where objects without specific coordinates are assumed to be evaluated at $\mathcal{P}$, so for example, $w^\beta = w^\beta(0,0)$. Parallel transport of $w^\beta(\lambda,0)$ a second time along $v^\alpha(\lambda,0)$ gives

$$
\begin{aligned}
\frac{\partial w^\beta}{d\sigma}(\lambda,0) &= -\left(v^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\right)\Big|_{(\lambda,0)} \\
w^\beta(\lambda,\sigma) &= w^\beta(\lambda,0) - \left(v^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\right)\Big|_{(\lambda,0)}\sigma \\
&= w^\beta(\lambda,0) - v^\alpha(\lambda,0)\, w^\mu(\lambda,0)\, \Gamma^\beta_{\ \mu\alpha}(\lambda,0)\,\sigma
\end{aligned}
$$

Expanding, keeping only up to second order,

$$
\begin{aligned}
w^\beta(\lambda,\sigma)_{uv} &= \left(w^\beta - u^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\lambda\right) - \left(v^\alpha - v^\sigma u^\gamma \Gamma^\alpha_{\ \gamma\sigma}\lambda\right)\left(w^\mu - u^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\lambda\right)\left(\Gamma^\beta_{\ \mu\alpha} + u^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\lambda\right)\sigma \\
&= w^\beta - u^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\lambda \\
&\quad + \left(-v^\alpha w^\mu + v^\alpha u^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\lambda + w^\mu v^\sigma u^\gamma \Gamma^\alpha_{\ \gamma\sigma}\lambda - u^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda} v^\sigma u^\gamma \Gamma^\alpha_{\ \gamma\sigma}\lambda^2\right)\left(\Gamma^\beta_{\ \mu\alpha}\sigma + u^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\lambda\sigma\right) \\
&= w^\beta - u^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\lambda - v^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\sigma \\
&\quad + \left(v^\alpha u^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\Gamma^\beta_{\ \mu\alpha} + w^\mu v^\sigma u^\gamma \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \mu\alpha} - v^\alpha w^\mu u^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\lambda\sigma
\end{aligned}
$$

Now repeat this calculation, but transporting along v^α first, then u^α. This looks just the same, but with interchange of u^α with v^α, and of σ with λ:

$$
\begin{aligned}
w^\beta(\lambda,\sigma)_{vu} &= w^\beta - v^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\sigma - u^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\lambda \\
&\quad + \left(u^\alpha v^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\Gamma^\beta_{\ \mu\alpha} + w^\mu u^\sigma v^\gamma \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \mu\alpha} - u^\alpha w^\mu v^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\sigma\lambda
\end{aligned}
$$

The difference in these expressions gives the change in w^β around the full loop,

$$
\begin{aligned}
dw^\beta &= w^\beta - u^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\lambda - v^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\sigma \\
&\quad + \left(v^\alpha u^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\Gamma^\beta_{\ \mu\alpha} + w^\mu v^\sigma u^\gamma \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \mu\alpha} - v^\alpha w^\mu u^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\lambda\sigma \\
&\quad - w^\beta + v^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\sigma + u^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\lambda \\
&\quad - \left(u^\alpha v^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\Gamma^\beta_{\ \mu\alpha} + w^\mu u^\sigma v^\gamma \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \mu\alpha} - u^\alpha w^\mu v^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\sigma\lambda \\
&= \left(u^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha} - u^\nu w^\rho \Gamma^\beta_{\ \rho\nu}\right)\lambda + \left(v^\nu w^\rho \Gamma^\beta_{\ \rho\nu} - v^\alpha w^\mu \Gamma^\beta_{\ \mu\alpha}\right)\sigma \\
&\quad + \left(v^\alpha u^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\Gamma^\beta_{\ \mu\alpha} + w^\mu v^\sigma u^\gamma \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \mu\alpha} - v^\alpha w^\mu u^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\lambda\sigma \\
&\quad - \left(u^\alpha v^\lambda w^\tau \Gamma^\mu_{\ \tau\lambda}\Gamma^\beta_{\ \mu\alpha} + w^\mu u^\sigma v^\gamma \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \mu\alpha} - u^\alpha w^\mu v^\nu \partial_\nu \Gamma^\beta_{\ \mu\alpha}\right)\sigma\lambda \\
&= w^\rho v^\sigma u^\gamma \left(\Gamma^\mu_{\ \rho\gamma}\Gamma^\beta_{\ \mu\sigma} + \Gamma^\alpha_{\ \gamma\sigma}\Gamma^\beta_{\ \rho\alpha} - \partial_\gamma \Gamma^\beta_{\ \rho\sigma}\right)\lambda\sigma \\
&\quad - w^\rho u^\gamma v^\sigma \left(\Gamma^\mu_{\ \rho\sigma}\Gamma^\beta_{\ \mu\gamma} + \Gamma^\alpha_{\ \sigma\gamma}\Gamma^\beta_{\ \rho\alpha} - \partial_\sigma \Gamma^\beta_{\ \rho\gamma}\right)\sigma\lambda
\end{aligned}
$$

Collecting terms,

$$
\begin{aligned}
dw^\beta &= w^\rho v^\sigma u^\gamma \left(\partial_\sigma \Gamma^\beta_{\ \rho\gamma} - \partial_\gamma \Gamma^\beta_{\ \rho\sigma} + \Gamma^\beta_{\ \mu\sigma}\Gamma^\mu_{\ \rho\gamma} - \Gamma^\beta_{\ \mu\gamma}\Gamma^\mu_{\ \rho\sigma} + \left(\Gamma^\alpha_{\ \gamma\sigma} - \Gamma^\alpha_{\ \sigma\gamma}\right)\Gamma^\beta_{\ \rho\alpha}\right)\sigma\lambda \\
dw^\beta &= w^\rho \left(\partial_\sigma \Gamma^\beta_{\ \rho\gamma} - \partial_\gamma \Gamma^\beta_{\ \rho\sigma} + \Gamma^\beta_{\ \mu\sigma}\Gamma^\mu_{\ \rho\gamma} - \Gamma^\beta_{\ \mu\gamma}\Gamma^\mu_{\ \rho\sigma}\right)\sigma\lambda v^\sigma u^\gamma
\end{aligned}
$$

This equation defines the curvature for any two directions u^α, v^β. Give a name to the term in parentheses:

$$R^\beta_{\ \rho\gamma\sigma} \equiv \Gamma^\beta_{\ \rho\gamma,\sigma} - \Gamma^\beta_{\ \rho\sigma,\gamma} + \Gamma^\beta_{\ \mu\sigma}\Gamma^\mu_{\ \rho\gamma} - \Gamma^\beta_{\ \mu\gamma}\Gamma^\mu_{\ \rho\sigma}$$

Then the relationship becomes

$$dw^\beta = w^\rho R^\beta_{\ \rho\gamma\sigma}\sigma\lambda u^\gamma v^\sigma$$

notice from its definition that $R^\beta_{\ \rho\gamma\sigma}$ is antisymmetric on its final two indices,

$$R^\beta_{\ \rho\gamma\sigma} = -R^\beta_{\ \rho\sigma\gamma}$$

We may use this property to produce an area element,

$$\begin{aligned} R^\beta_{\ \rho\gamma\sigma}u^\gamma v^\sigma &= \frac{1}{2}\left(R^\beta_{\ \rho\gamma\sigma}u^\gamma v^\sigma + R^\beta_{\ \rho\gamma\sigma}u^\gamma v^\sigma\right) \\ &= \frac{1}{2}\left(R^\beta_{\ \rho\gamma\sigma}u^\gamma v^\sigma - R^\beta_{\ \rho\sigma\gamma}u^\gamma v^\sigma\right) \\ &= \frac{1}{2}\left(R^\beta_{\ \rho\gamma\sigma}u^\gamma v^\sigma - R^\beta_{\ \rho\gamma\sigma}u^\sigma v^\gamma\right) \\ &= R^\beta_{\ \rho\gamma\sigma}\frac{1}{2}\left(u^\gamma v^\sigma - u^\sigma v^\gamma\right) \end{aligned}$$

In our case, we may take u^α and v^β to be unit vectors, with the magnitude of the area given by $\sigma\lambda$. Therefore, we define the infinitesimal area element

$$dA^{\gamma\sigma} \equiv \frac{1}{2}\sigma\lambda\left(u^\gamma v^\sigma - u^\sigma v^\gamma\right)$$

We now have

$$\begin{aligned} dw^\beta &= w^\rho R^\beta_{\ \rho\gamma\sigma}dA^{\gamma\sigma} \\ &= w^\rho dT^\beta_\rho \end{aligned}$$

where $dT^\beta_\rho = R^\beta_{\ \rho\gamma\sigma}dA^{\gamma\sigma}$. Then *Riemann curvature tensor* is the transformation per unit area is

$$\begin{aligned} R^\alpha_{\ \beta\mu\nu} &= \frac{dT^\alpha_\beta}{dA^{\mu\nu}} \\ &= \Gamma^\beta_{\ \rho\gamma,\sigma} - \Gamma^\beta_{\ \rho\sigma,\gamma} + \Gamma^\beta_{\ \mu\sigma}\Gamma^\mu_{\ \rho\gamma} - \Gamma^\beta_{\ \mu\gamma}\Gamma^\mu_{\ \rho\sigma} \end{aligned}$$

This gives the form of the curvature tensor in terms of the connection and its first derivatives.

4 Curvature as a tensor

Since all elements of this construction are defined geometrically from within the manifold, $R^\alpha_{\ \mu\beta\nu}$ is intrinsic to the manifold, and therefore independent of our choice of coordinates. To see this explicitly, we could recompute the same construction in different coordinates. Since the entire construction is perturbative, we would find the components of $R^\alpha_{\ \mu\beta\nu}$ changing linearly and homogeneously in the transformation matrix. However, there is an easier way to prove that $R^\alpha_{\ \mu\beta\nu}$ is a tensor. Consider two covariant derivatives of the vector w^α,

$$\begin{aligned} D_\mu D_\nu w^\alpha &= D_\mu\left(\partial_\nu w^\alpha + w^\beta\Gamma^\alpha_{\ \beta\nu}\right) \\ &= \partial_\mu\left(\partial_\nu w^\alpha + w^\beta\Gamma^\alpha_{\ \beta\nu}\right) + \left(\partial_\nu w^\rho + w^\beta\Gamma^\rho_{\ \beta\nu}\right)\Gamma^\alpha_{\ \rho\mu} - \left(\partial_\rho w^\alpha + w^\beta\Gamma^\alpha_{\ \rho\nu}\right)\Gamma^\rho_{\ \nu\mu} \end{aligned}$$

Because we use covariant derivatives, this object is necessarily a tensor. Now take the derivatives in the opposite order and subtract, giving the commutator. This is also a necessarily a tensor,

$$\begin{aligned}
[D_\mu, D_\nu]\, w^\alpha &= D_\mu D_\nu w^\alpha - D_\nu D_\mu w^\alpha \\
&= \partial_\mu \partial_\nu w^\alpha + \partial_\mu w^\beta \Gamma^\alpha_{\ \beta\nu} + w^\beta \Gamma^\alpha_{\ \beta\nu,\mu} + \left(\partial_\nu w^\rho + w^\beta \Gamma^\rho_{\ \beta\nu}\right) \Gamma^\alpha_{\ \rho\mu} - \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right) \Gamma^\rho_{\ \nu\mu} \\
&\quad - \partial_\nu \partial_\mu w^\alpha - \partial_\nu w^\beta \Gamma^\alpha_{\ \beta\mu} - w^\beta \Gamma^\alpha_{\ \beta\mu,\nu} - \left(\partial_\mu w^\rho + w^\beta \Gamma^\rho_{\ \beta\mu}\right) \Gamma^\alpha_{\ \rho\nu} + \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right) \Gamma^\rho_{\ \mu\nu} \\
&= \partial_\mu \partial_\nu w^\alpha - \partial_\nu \partial_\mu w^\alpha + \partial_\mu w^\beta \Gamma^\alpha_{\ \beta\nu} + \partial_\nu w^\rho \Gamma^\alpha_{\ \rho\mu} - \partial_\nu w^\beta \Gamma^\alpha_{\ \beta\mu} - \partial_\mu w^\rho \Gamma^\alpha_{\ \rho\nu} \\
&\quad - \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right) \Gamma^\rho_{\ \nu\mu} + \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right) \Gamma^\rho_{\ \mu\nu} \\
&\quad + w^\beta \Gamma^\alpha_{\ \beta\nu,\mu} + w^\beta \Gamma^\rho_{\ \beta\nu} \Gamma^\alpha_{\ \rho\mu} - w^\beta \Gamma^\alpha_{\ \beta\mu,\nu} - w^\beta \Gamma^\rho_{\ \beta\mu} \Gamma^\alpha_{\ \rho\nu} \\
&= w^\beta R^\alpha_{\ \beta\nu\mu}
\end{aligned}$$

Since the result is w^β properly contracted with $R^\alpha_{\ \beta\nu\mu}$, and w^β is a tensor, $R^\alpha_{\ \beta\nu\mu}$ must also be a tensor.

5 Symmetries of the curvature tensor

We have already seen that since parallel transport preserves lengths, $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$, and because it describes a transformation per unit area, $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$. Explicitly, from the definition,

$$\begin{aligned}
R^\alpha_{\ \beta\mu\nu} &= \Gamma^\alpha_{\ \beta\mu,\nu} - \Gamma^\alpha_{\ \beta\nu,\mu} + \Gamma^\alpha_{\ \sigma\nu}\Gamma^\sigma_{\ \beta\mu} - \Gamma^\alpha_{\ \sigma\mu}\Gamma^\sigma_{\ \beta\nu} \\
&= -\Gamma^\alpha_{\ \beta\nu,\mu} + \Gamma^\alpha_{\ \beta\mu,\nu} - \Gamma^\alpha_{\ \sigma\mu}\Gamma^\sigma_{\ \beta\nu} + \Gamma^\alpha_{\ \sigma\nu}\Gamma^\sigma_{\ \beta\mu} \\
&= -\left(\Gamma^\alpha_{\ \beta\nu,\mu} - \Gamma^\alpha_{\ \beta\mu,\nu} + \Gamma^\alpha_{\ \sigma\mu}\Gamma^\sigma_{\ \beta\nu} - \Gamma^\alpha_{\ \sigma\nu}\Gamma^\sigma_{\ \beta\mu}\right) \\
&= -R^\alpha_{\ \beta\nu\mu}
\end{aligned}$$

Another symmetry follows if we totally antisymmetrize the final three indices,

$$\begin{aligned}
R^\alpha_{\ [\beta\mu\nu]} &= \frac{1}{3}\left(R^\alpha_{\ \beta\mu\nu} + R^\alpha_{\ \mu\nu\beta} + R^\alpha_{\ \nu\beta\mu}\right) \\
3R^\alpha_{\ [\beta\mu\nu]} &= \Gamma^\alpha_{\ \beta\mu,\nu} - \Gamma^\alpha_{\ \beta\nu,\mu} + \Gamma^\alpha_{\ \sigma\nu}\Gamma^\sigma_{\ \beta\mu} - \Gamma^\alpha_{\ \sigma\mu}\Gamma^\sigma_{\ \beta\nu} \\
&\quad + \Gamma^\alpha_{\ \mu,\nu\beta} - \Gamma^\alpha_{\ \mu\beta,\nu} + \Gamma^\alpha_{\ \sigma\beta}\Gamma^\sigma_{\ \mu\nu} - \Gamma^\alpha_{\ \sigma\nu}\Gamma^\sigma_{\ \mu\beta} \\
&\quad + \Gamma^\alpha_{\ \nu,\beta\mu} - \Gamma^\alpha_{\ \nu\mu,\beta} + \Gamma^\alpha_{\ \sigma\mu}\Gamma^\sigma_{\ \nu\beta} - \Gamma^\alpha_{\ \sigma\beta}\Gamma^\sigma_{\ \nu\mu} \\
&= \left(\Gamma^\alpha_{\ \beta\mu} - \Gamma^\alpha_{\ \mu\beta}\right)_{,\nu} + \left(\Gamma^\alpha_{\ \nu\beta} - \Gamma^\alpha_{\ \beta\nu}\right)_{,\mu} + \left(\Gamma^\alpha_{\ \mu\nu} - \Gamma^\alpha_{\ \nu\mu}\right)_{,\beta} \\
&\quad + \Gamma^\alpha_{\ \sigma\nu}\left(\Gamma^\sigma_{\ \beta\mu} - \Gamma^\sigma_{\ \mu\beta}\right) + \Gamma^\alpha_{\ \sigma\beta}\left(\Gamma^\sigma_{\ \mu\nu} - \Gamma^\sigma_{\ \nu\mu}\right) \\
&\quad + \Gamma^\alpha_{\ \sigma\mu}\left(\Gamma^\sigma_{\ \nu\beta} - \Gamma^\sigma_{\ \beta\nu}\right) \\
&= 0
\end{aligned}$$

The sum vanishes identically because of the symmetry of $\Gamma^\alpha_{\ \mu\nu}$. This condition ultimately arises because the connection may be written in terms of the metric. It is called the first Bianchi identity.

Finally, consider

$$\begin{aligned}
R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} &= R_{\alpha\beta\mu\nu} - \left(-R_{\mu\alpha\beta\nu} - R_{\mu\beta\nu\alpha}\right) \\
&= R_{\alpha\beta\mu\nu} - R_{\alpha\mu\beta\nu} - R_{\beta\mu\nu\alpha} \\
&= R_{\alpha\beta\mu\nu} - \left(-R_{\alpha\beta\nu\mu} - R_{\alpha\nu\mu\beta}\right) - \left(-R_{\beta\alpha\mu\nu} - R_{\beta\nu\alpha\mu}\right) \\
&= R_{\alpha\beta\mu\nu} + R_{\alpha\beta\nu\mu} + R_{\alpha\nu\mu\beta} + R_{\beta\alpha\mu\nu} + R_{\beta\nu\alpha\mu} \\
&= -R_{\alpha\beta\mu\nu} + R_{\alpha\nu\mu\beta} + R_{\beta\nu\alpha\mu}
\end{aligned}$$

$$\begin{aligned}
&= (-R_{\alpha\beta\mu\nu} - R_{\alpha\nu\beta\mu}) + R_{\beta\nu\alpha\mu} \\
&= R_{\alpha\mu\nu\beta} + R_{\beta\nu\alpha\mu} \\
&= -R_{\alpha\mu\beta\nu} + R_{\beta\nu\alpha\mu}
\end{aligned}$$

Now interchange the names within each pair, $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$. On the left, this gives two minus signs, but on the right only one:

$$\begin{aligned}
R_{\beta\alpha\nu\mu} - R_{\nu\mu\beta\alpha} &= -R_{\beta\nu\alpha\mu} + R_{\alpha\mu\beta\nu} \\
(-1)^2 (R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}) &= +R_{\alpha\mu\beta\nu} - R_{\beta\nu\alpha\mu} \\
&= -(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta})
\end{aligned}$$

This difference therefore vanishes, and we have symmetry under interchange of the pairs,

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

Summarizing, we have the following symmetries of the Riemann curvature tensor:

$$\begin{aligned}
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha[\beta\mu\nu]} &= 0
\end{aligned}$$

We can count the independent components by using these symmetries. Because of the antisymmetry on $\alpha\beta$, there are only $\frac{4\cdot 3}{2} = 6$ independent values for this pair of indices. The same counting holds for the final pair, $\mu\nu$. Since we have symmetry in these pairs,

$$R_{[\alpha\beta][\mu\nu]} = R_{[\mu\nu][\alpha\beta]}$$

we may think of $R_{[\alpha\beta][\mu\nu]}$ as a 6×6 symmetric matrix, which will have $\frac{6\cdot 7}{2} = 21$ independent components. This makes use of the first three symmetries.

To use the final symmetry, note that the three final indices must differ from one another, so there are only possible four cases,

$$\begin{aligned}
R_{0[\alpha\beta\mu]} &= 0 \\
R_{1[\alpha\beta\mu]} &= 0 \\
R_{2[\alpha\beta\mu]} &= 0 \\
R_{3[\alpha\beta\mu]} &= 0
\end{aligned}$$

Now suppose one of $\alpha\beta\mu$ is the same as the first index, for example,

$$\begin{aligned}
R_{1123} + R_{1312} + R_{1231} &= R_{1312} + R_{1231} \\
&= R_{1312} - R_{1213} \\
&= 0
\end{aligned}$$

Then the vanishing is automatic using the previous three symmetries and there is no additional constraint. Therefore, to get any new condition, all four indices must differ. But then, notice that

$$\begin{aligned}
R_{0123} &= -R_{1023} \\
&= R_{2310} \\
&= -R_{3210}
\end{aligned}$$

so that once we have the condition with 0 in the first position, the other three possibilities follow automatically. There is therefore only one condition from the fourth symmetry,

$$R_{0123} + R_{0312} + R_{0231} = 0$$

reducing the number of degrees of freedom of the Riemann curvature tensor to 20.

6 Ricci tensor and Ricci scalar

Because of the symmetries, there is only one independent contraction of $R^{\alpha}_{\ \beta\mu\nu}$. We define the Ricci tensor,

$$R_{\mu\nu} \equiv R^{\alpha}_{\ \mu\alpha\nu}$$

Because of the symmetry between pairs, we have

$$\begin{aligned} R_{\mu\nu} &\equiv R^{\alpha}_{\ \mu\alpha\nu} \\ &= g^{\alpha\beta} R_{\alpha\mu\beta\nu} \\ &= g^{\alpha\beta} R_{\beta\nu\alpha\mu} \\ &= R^{\alpha}_{\ \nu\alpha\mu} \\ &= R_{\nu\mu} \end{aligned}$$

so the Ricci tensor is symmetric.

Exercise: Prove that any other contraction of the curvature is either zero, or a multiple of the Ricci tensor.

We also define the Ricci scalar, given by taking contracting the Ricci tensor,

$$R = g^{\mu\nu} R_{\mu\nu}$$

It is possible to decompose the full Riemann curvature into a traceless part, called the Weyl curvature, and combinations of the Ricci tensor and Ricci scalar, but we will not need this now.

7 The second Bianchi identity and the Einstein equation

We have already seen the first Bianchi identity,

$$R_{\alpha[\beta\mu\nu]} = 0$$

This is an integrability condition that guarantees that the connection may be written in terms of derivatives of the metric. There is a second integrability condition, called the second Bianchi identity, guaranteeing that the curvature may be written in terms of a connection. The second Bianchi identity is

$$R^{\alpha}_{\ \beta[\mu\nu;\sigma]} = 0$$

To prove this, first consider antisymmetrizing a double commutator:

$$\begin{aligned} \left[D_\beta, \left[D_\mu, D_\nu\right]\right] w^\alpha &= \left(D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu - D_\mu D_\nu D_\beta + D_\nu D_\mu D_\beta\right) w^\alpha \\ 3\left[D_{[\beta}, \left[D_\mu, D_{\nu]}\right]\right] &= \left[D_\beta, \left[D_\mu, D_\nu\right]\right] + \left[D_\mu, \left[D_\nu, D_\beta\right]\right] + \left[D_\nu, \left[D_\beta, D_\mu\right]\right] \\ &= D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu - D_\mu D_\nu D_\beta + D_\nu D_\mu D_\beta \\ &\quad + D_\mu D_\nu D_\beta - D_\mu D_\beta D_\nu - D_\nu D_\beta D_\mu + D_\beta D_\nu D_\mu \\ &\quad + D_\nu D_\beta D_\mu - D_\nu D_\mu D_\beta - D_\beta D_\mu D_\nu + D_\mu D_\beta D_\nu \\ &= D_\beta D_\mu D_\nu - D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu + D_\beta D_\nu D_\mu \\ &\quad + D_\mu D_\nu D_\beta - D_\mu D_\nu D_\beta - D_\mu D_\beta D_\nu + D_\mu D_\beta D_\nu \\ &\quad - D_\nu D_\beta D_\mu + D_\nu D_\beta D_\mu - D_\nu D_\mu D_\beta + D_\nu D_\mu D_\beta \\ &= 0 \end{aligned}$$

However, we may also write this as

$$\begin{aligned}
0 &= 3\left[D_{[\beta},\left[D_\mu,D_{\nu]}\right]\right]w^\alpha \\
&= D_\beta D_\mu D_\nu - D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu + D_\beta D_\nu D_\mu \\
&\quad + D_\mu D_\nu D_\beta - D_\mu D_\nu D_\beta - D_\mu D_\beta D_\nu + D_\mu D_\beta D_\nu \\
&\quad - D_\nu D_\beta D_\mu + D_\nu D_\beta D_\mu - D_\nu D_\mu D_\beta + D_\nu D_\mu D_\beta \\
&= D_\beta\left(D_\mu D_\nu - D_\nu D_\mu\right)w^\alpha + D_\beta\left(D_\nu D_\mu - D_\mu D_\nu\right)w^\alpha \\
&\quad + D_\mu\left(D_\nu D_\beta - D_\beta D_\nu\right)w^\alpha + D_\mu\left(D_\beta D_\nu - D_\nu D_\beta\right)w^\alpha \\
&\quad + D_\nu\left(D_\mu D_\beta - D_\beta D_\mu\right)w^\alpha + D_\nu\left(D_\beta D_\mu - D_\mu D_\beta\right)w^\alpha \\
&= D_\beta\left(w^\rho R^\alpha_{\ \rho\mu\nu}\right) + D_\beta\left(w^\rho R^\alpha_{\ \rho\nu\mu}\right) + D_\mu\left(w^\rho R^\alpha_{\ \rho\nu\beta}\right) \\
&\quad + D_\mu\left(w^\rho R^\alpha_{\ \rho\beta\nu}\right) + D_\nu\left(w^\rho R^\alpha_{\ \rho\mu\beta}\right) + D_\nu\left(w^\rho R^\alpha_{\ \rho\beta\mu}\right) \\
&= w^\rho R^\alpha_{\ \rho[\mu\nu;\beta]} \\
&\quad + \left(R^\alpha_{\ \rho\mu\nu} + R^\alpha_{\ \rho\nu\mu}\right)D_\beta w^\rho + \left(R^\alpha_{\ \rho\nu\beta} + R^\alpha_{\ \rho\beta\nu}\right)D_\mu w^\rho + \left(R^\alpha_{\ \rho\mu\beta} + R^\alpha_{\ \rho\beta\mu}\right)D_\nu w^\rho \\
&= w^\rho R^\alpha_{\ \rho[\mu\nu;\beta]}
\end{aligned}$$

and since w^α is arbitrary, we have the second Bianchi identity.

The second Bianchi identity is important for general relativity because of its contractions. First, expand the identity and contract on $\alpha\mu$,

$$\begin{aligned}
0 &= R^\alpha_{\ \beta\mu\nu;\sigma} + R^\alpha_{\ \beta\nu\sigma;\mu} + R^\alpha_{\ \beta\sigma\mu;\nu} \\
0 &= R^\alpha_{\ \beta\alpha\nu;\sigma} + R^\alpha_{\ \beta\nu\sigma;\alpha} + R^\alpha_{\ \beta\sigma\alpha;\nu}
\end{aligned}$$

Using the definition of the Ricci tensor and the antisymmetry of the Riemann tensor,

$$0 = R_{\beta\nu;\sigma} + R^\alpha_{\ \beta\nu\sigma;\alpha} - R_{\beta\sigma;\nu}$$

Now contract on $\beta\sigma$ using the metric,

$$\begin{aligned}
0 &= g^{\beta\sigma}R_{\beta\nu;\sigma} + g^{\beta\sigma}R^\alpha_{\ \beta\nu\sigma;\alpha} - g^{\beta\sigma}R_{\beta\sigma;\nu} \\
&= \left(g^{\beta\sigma}R_{\beta\nu}\right)_{;\sigma} + g^{\beta\sigma}R_{\beta\ \sigma\nu;\alpha}^{\ \alpha} - \left(g^{\beta\sigma}R_{\beta\sigma}\right)_{;\nu} \\
&= R^\sigma_{\ \nu;\sigma} + R^\alpha_{\ \nu;\alpha} - R_{;\nu}
\end{aligned}$$

We may write this as a divergence,

$$\begin{aligned}
0 &= R^\alpha_{\ \nu;\alpha} - \frac{1}{2}R_{;\nu} \\
0 &= D_\alpha\left(R^\alpha_{\ \nu} - \frac{1}{2}\delta^\alpha_\nu R\right)
\end{aligned}$$

or, raising an index,

$$D_\alpha\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = 0$$

We define the Einstein tensor,

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$$

and have now shown that it has vanishing divergence. Since both the Ricci tensor and the metric are both symmetric, we have

$$\begin{aligned}
G^{\alpha\beta} &= G^{\beta\alpha} \\
D_\beta G^{\alpha\beta} &= 0
\end{aligned}$$

These are precisely the properties we require of the energy-momentum tensor, $T^{\alpha\beta}$. It can be shown that $G^{\alpha\beta}$ is the only tensor linear in components of the Riemann curvature tensor to have these properties.

Reasoning that it is the presence of energy that leads to curvature, the only candidate equation consistent with the properties of $T^{\alpha\beta}$ and linear in the curvature (hence, a second order differential equation for the metric) is

$$G^{\alpha\beta} = \kappa T^{\alpha\beta}$$

This is the Einstein equation for general relativity.