Problems in relativistic dynamics

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1 Compton scattering

Let a photon impinge on an electron and scatter, making an angle θ with its original direction. At the same time, the electron, which was initially at rest, also recoils. Show that the change in wavelength of the photon is given by the Compton formula,

$$\tilde{\lambda} - \lambda = \frac{h}{mc}(1 - \cos\theta)$$

Hints: Let the initial photon have 4-momentum $k^\alpha = \hbar\left(\frac{\omega}{c}, \mathbf{k}\right)$ with the electron initially at rest so that $p^\alpha = (mc, \mathbf{0})$. Let the final photon momentum be $\tilde{k}^\alpha = \hbar\left(\frac{\tilde{\omega}}{c}, \tilde{\mathbf{k}}\right)$ and the final electron have $p^\alpha = \gamma m(c, \mathbf{v})$. Write the conservation of momentum

$$k^\alpha + p^\alpha = \tilde{k}^\alpha + \tilde{p}^\alpha$$

and rewrite it as $\tilde{p}^\alpha = k^\alpha + p^\alpha - \tilde{k}^\alpha$. That way, when you take the norm of both sides, $\tilde{p}^\alpha\tilde{p}_\alpha$, the γ and $\mathbf{v}$ won't appear in the equation. The square of the right side, $\left(k^\alpha + p^\alpha - \tilde{k}^\alpha\right)\left(k_\alpha + p_\alpha - \tilde{k}_\alpha\right)$ is not difficult. Notice that $\omega = |\mathbf{k}|\, c$ and $|\mathbf{k}| = \frac{2\pi}{\lambda}$.

2 Photon decay

Address the question of whether a photon can decay into three particles: an electron, a positron, and a photon. In particular, determine whether the decay is possible. You may work in any frame of reference, but I recommend the center of mass frame of the electron-positron pair.

3 Constant acceleration

A particle experiences a constant acceleration, β, in the x-direction, in its instantaneous rest frame. That means that in that frame, its acceleration 4-vector is simply $a^\alpha = (0, \beta, 0, 0)$, but the frame in which this holds is constantly changing. However, we may compute

$$a^\alpha a_\alpha = \beta^2$$

and this relation is independent of the frame of reference. To describe the motion, we must integrate twice,

$$\begin{aligned} a^\alpha &= \frac{du^\alpha}{d\tau} \\ u^\alpha &= \frac{dx^\alpha}{d\tau} \end{aligned}$$

and this will give us $x^\alpha(\tau)$. Keep the problem two dimensional (t, x) by choosing the motion to lie purely in the x-direction. Then we have three algebraic relations,

$$\begin{aligned} a^\alpha a_\alpha &= -\left(a^0\right)^2 + \left(a^1\right)^2 = \beta^2 \\ u^\alpha a_\alpha &= -a^0 u^0 + a^1 u^1 = 0 \\ u^\alpha u_\beta &= -\left(u^0\right)^2 + \left(u^1\right)^2 = -c^2 \end{aligned}$$

Consider the $\alpha = 0$ equation, $a^0 = \frac{du^0}{d\tau}$. Use the algebraic relations to write a^0 as a function of u^0, and you will be able to integrate.