Problems in general relativity

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1. Compute the Laplacian of a function, $\nabla^2 f = g^{ij} D_i D_j f$, in polar coordinates.
2. Compute the Laplacian of a function, $\nabla^2 f = g^{ij} D_i D_j f$, in spherical coordinates.
3. The metric in spherical coordinates is

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$$

   Compute all nonvanishing Christoffel symbols, $\Gamma^i{}_{jk}$.

4. Let the metric of spacetime be nearly flat, and of the diagonal form

$$g_{\mu\nu} = \begin{pmatrix} -c^2 + \phi & & & \\ & 1 + \lambda\phi & & \\ & & 1 + \lambda\phi & \\ & & & 1 + \lambda\phi \end{pmatrix}$$

   where $\phi = \phi(r)$ and $r = \sqrt{x^2 + y^2 + z^2}$. In the notes we found that with $\lambda = 0$ the spatial components of the geodesic equation reproduce Newton's law of universal gravitation for a suitable choice of ϕ. However, the time component of the geodesic equation was off by a factor of 2. Repeat the calculation for nonzero λ and determine whether there is a choice for the constant λ and the function ϕ which also gives the correct rate of change of energy in the nonrelativistic limit.

5. Find all nonvanishing components of the connection $\Gamma^i{}_{jk}$ for a 3-dimensional space described by the line element

$$ds^2 = dx^2 + 2x^2 dx dy + y^2 dz^2$$

   The metric is then $g_{ij} = \begin{pmatrix} 1 & x^2 & 0 \\ x^2 & 0 & \\ 0 & 0 & y^2 \end{pmatrix}$. What is the inverse metric? Be careful when raising the index to Γ^i_{jk} from Γ_{ijk}!

6. Explore the difference between a torus and the surface of a doughnut. We may define a torus as a cylinder with the ends identified. Find the connection on such a cylinder, $\rho = R = constant$ so that

$$ds^2 = \rho_0^2 d\varphi^2 + dz^2$$

   By contrast, we may define a doughnut shape by the constraint

$$z^2 + (R - \rho)^2 = d^2$$

   with $R > d$. Use the constraint to find the metric, then derive the components of the connection.