Parallel transport and curvature

February 22, 2015

1 Parallel transport around a closed loop

Consider parallel transport around a loop at constant θ on S^2 with the usual metric,

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$$

or, as a matrix,

$$g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

The only nonvanishing derivative is $g_{22,1} = 2R^2 \sin\theta\cos\theta$.

1.1 The connection

First we find the connection. Because the only nonvanishing

$$\begin{aligned}
\Gamma_{221} &= \Gamma_{212} \\
&= \frac{1}{2}\left(g_{22,1} + g_{21,2} - g_{21,2}\right) \\
&= \frac{1}{2}\left(g_{22,1} + 0 - 0\right) = R^2\sin\theta\cos\theta \\
&= R^2\sin\theta\cos\theta \\
\Gamma_{122} &= \frac{1}{2}\left(g_{12,2} + g_{12,2} - g_{22,1}\right) \\
&= -R^2\sin\theta\cos\theta
\end{aligned}$$

and therefore

$$\begin{aligned}
\Gamma^2_{\ 12} &= \Gamma^2_{\ 21} = \frac{\cos\theta}{\sin\theta} \\
\Gamma^1_{\ 22} &= -\sin\theta\cos\theta
\end{aligned}$$

1.2 The parallel transport equations

Let the initial vector be $v^a = (a, b)$ and carry out parallel transport along $u^a = (0, c)$. Then we set $u^b D_b v^a = 0$ where

$$u^b D_b v^a = u^b \partial_b v^a + u^b v^c \Gamma^a_{\ cb}$$

For the θ component,

$$\begin{aligned}
0 &= u^b \partial_b v^1 + u^b v^c \Gamma^1_{\ cb} \\
&= c\frac{\partial v^1}{\partial\varphi} + cv^2\Gamma^1_{\ 22} \\
&= c\frac{\partial v^1}{\partial\varphi} - cv^2\sin\theta\cos\theta
\end{aligned}$$

and for φ,

$$\begin{aligned}
0 &= u^b \partial_b v^2 + u^b v^c \Gamma^2_{cb} \\
&= c\frac{\partial v^2}{\partial \varphi} + cv^1 \Gamma^2_{12} \\
&= c\frac{\partial v^2}{\partial \varphi} + cv^1 \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

giving a pair of equations,

$$\begin{aligned}
\frac{\partial v^1}{\partial \varphi} &= v^2 \sin\theta\cos\theta \\
\frac{\partial v^2}{\partial \varphi} &= -v^1 \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

where $\theta = constant$. Differentiate the first again,

$$\begin{aligned}
\frac{\partial^2 v^1}{\partial \varphi^2} &= \frac{\partial v^2}{\partial \varphi} \sin\theta\cos\theta \\
&= -v^1 \frac{\cos\theta}{\sin\theta} \sin\theta\cos\theta \\
\frac{\partial^2 v^1}{\partial \varphi^2} + v^1 \cos^2\theta &= 0
\end{aligned}$$

so that we immediately have

$$v^1 = A\cos(\varphi\cos\theta) + B\sin(\varphi\cos\theta)$$

Differentiating to find v^2,

$$\begin{aligned}
\frac{\partial v^1}{\partial \varphi} &= v^2 \sin\theta\cos\theta \\
(-A\sin(\varphi\cos\theta) + B\cos(\varphi\cos\theta))\cos\theta &= v^2 \sin\theta\cos\theta
\end{aligned}$$

so

$$v^2 = \frac{1}{\sin\theta}(-A\sin(\varphi\cos\theta) + B\cos(\varphi\cos\theta))$$

At $\varphi = 0$ we impose the initial condition, $\left(v^1, v^2\right) = (a, b)$, so

$$\begin{aligned}
a &= A \\
b &= \frac{B}{\sin\theta}
\end{aligned}$$

and therefore,

$$\begin{aligned}
v^1 &= a\cos(\varphi\cos\theta) + b\sin\theta\sin(\varphi\cos\theta) \\
v^2 &= \left(-\frac{a}{\sin\theta}\sin(\varphi\cos\theta) + b\cos(\varphi\cos\theta)\right)
\end{aligned}$$

1.3 Rotation of v^a after a full circuit

After a 2π circuit,

$$\begin{aligned}
v^1 &= a\cos(2\pi\cos\theta) + b\sin\theta\sin(2\pi\cos\theta) \\
v^2 &= \left(-\frac{a}{\sin\theta}\sin(2\pi\cos\theta) + b\cos(2\pi\cos\theta)\right)
\end{aligned}$$

the cosine of the angle made with the original vector is then,

$$\begin{aligned}\cos\delta &= \frac{g_{ij}v^i_{initial}v^j_{final}}{g_{ij}v^i_{initial}v^j_{initial}} \\ &= \frac{R^2\left(a^2\cos(2\pi\cos\theta) + ab\sin\theta\sin(2\pi\cos\theta) - ab\sin\theta\sin(2\pi\cos\theta) + b^2\sin^2\theta\cos(2\pi\cos\theta)\right)}{R^2\left(a^2 + b^2\sin^2\theta\right)} \\ &= \frac{\left(a^2 + b^2\sin^2\theta\right)\cos(2\pi\cos\theta)}{a^2 + b^2\sin^2\theta} \\ &= \cos(2\pi\cos\theta)\end{aligned}$$

Notice that the result is independent of the original vector. It is telling us something about the shape of the space.

Near $\theta = 0$, $\cos\theta$ is very close to 1 and the rotation is close to a full turn, $\delta \approx 2\pi$. As θ increases, $\delta = 2\pi\cos\theta$ decreases continuously. At $\theta = \frac{\pi}{4}$, $\delta = \frac{\pi}{\sqrt{2}}$, and at the equator, $\theta = \frac{\pi}{2}$, the cosine vanishes, $\cos\theta = 0$, and we have $\delta = 0$, so the vector is not rotated at all.

If we considered instead motion in a plane, the rotation of a vector is exactly 2π, so there is a difference of

$$\Delta = \delta_{circle} - \delta_{sphere} = 2\pi(1 - \cos\theta)$$

This is the deviation of rotation angle from the rotation in flat space. We will call this the *angular deficit* (other definitions, essentially equivalent, vary from this). Notice that this calculation depends only on solving the parallel transport equation, which depends only on our knowledge of the metric on S^2. We did not require any knowledge of the enveloping Euclidean space.

2 Rotation per unit area

Consider the rate of change of deviation of the rotation angle with respect to the enclosed area. The area is given by integrating the 2-dimensional volume element. Let $g \equiv det(g_{ij})$, then

$$\sqrt{g}d\theta d\varphi = R^2\sin\theta d\theta d\varphi$$

Notice that this also depends only on the metric. The area within a circle of proper radius $s = R\theta$ is

$$\begin{aligned}A(s) &= \int_0^\theta\int_0^{2\pi} R^2\sin\theta d\theta d\varphi \\ &= -2\pi R^2\cos\theta\Big|_0^\theta \\ &= 2\pi R^2(1 - \cos\theta)\end{aligned}$$

As we shrink the area to a point, the derivative of the angular deficit with respect to area remains finite,

$$\begin{aligned}d\Delta &= d(2\pi(1 - \cos\theta)) \\ &= 2\pi\sin\theta d\theta \\ dA &= 2\pi R^2\sin\theta d\theta\end{aligned}$$

so

$$\begin{aligned}\left.\frac{d\Delta}{dA}\right|_{\theta=0} &= \left.\frac{2\pi\sin\theta d\theta}{2\pi R^2\sin\theta d\theta}\right|_{\theta=0} \\ &= \frac{1}{R^2}\end{aligned}$$

This is a remarkable result. Using only the metric of the sphere, we have computed its radius. This means that there is geometric information about the *shape* of a space – how it appears to an "outside" observer, that can be determined from measurements made within the space. Consider our method. We move a vector by parallel transport, that is, so that infinitesimal displacement by infinitesimal displacement it does not rotate. We carry the vector in this way around a closed path and discover that it returns rotated. We compare this rotation to the area enclosed by the closed path, and in the limit of vanishingly small areas we get a measure of the curvature of the sphere.

We call this derivative the *curvature* of the 2-sphere.

3 Exercise

Compute the curvature of the parabolic surface $z = \frac{1}{2}a\rho^2$ at the origin.