More problems in general relativity

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1. The Schwarzschild solution, in r, t, θ, φ coordinates, diverges at $r = 0$ and $r = 2M$. We have seen that changing to Kruskal-Szekeres coordinate removes any pathology at $r = 2M$: this is just normal (curved) spacetime. The only meaning of the *event horizon* at $r = 2M$ is large scale, telling us about which events can be seen by which observers. On the other hand, the singularity at $r = 0$ is a failure of the geometry to exist at that point. To show this, it is sufficient to show the existence of a *scalar* that diverges. The individual components of vectors or tensors depend on our choice of coordinates, but a scalar such as $v^\alpha w_\alpha$ or $R^{\alpha\beta} R_{\alpha\beta}$ is coordinate invariant. The simplest scalars to compute are the Ricci scalar, R, and the square of the Ricci tensor, $R^{\alpha\beta} R_{\alpha\beta}$, but since the Schwarzschild geometry satisfies $R_{\alpha\beta} = 0$, these both vanish. Prove that $r = 0$ is a true singularity by showing that

$$\lim_{r \to 0} \left(R^{\beta}_{\ \alpha\rho\sigma} R^{\alpha}_{\ \beta\mu\nu} g^{\rho\mu} g^{\sigma\nu} \right) = \infty$$

   Notice that your result does not diverge at $r = 2M$. Showing one divergent scalar is enough to prove that a point is not part of the manifold, but proving that a point is regular requires all scalars to be regular – for these it is often easier to find coordinates that remove the problem. When you sum, you will get several equivalent terms, for example, both of

$$R^{1}_{\ 0\rho\sigma} R^{0}_{\ 1\mu\nu} g^{\rho\mu} g^{\sigma\nu}$$
$$R^{0}_{\ 1\rho\sigma} R^{1}_{\ 0\mu\nu} g^{\rho\mu} g^{\sigma\nu}$$

   occur in the sum.

2. By describing a 3-dimensional spherical surface embedded in a 4-dimensional space, we can find a way to write the metric of a constant curvature space. Let the 4-dimensional (Euclidean or Lorentzian) space have line element

$$ds^2 = \lambda dw^2 + dx^2 + dy^2 + dz^2$$

   where $\lambda = \pm 1$, and let the surface be described by

$$\lambda w^2 + x^2 + y^2 + z^2 = \lambda R^2$$

   Differentiating the constraint to the surface gives

$$2\lambda w dw + 2\mathbf{x} \cdot d\mathbf{x} = 0$$

   where $\mathbf{x} \cdot d\mathbf{x} = \delta_{ij} x^i dx^j$ is the usual Euclidean dot product. Then

$$\begin{aligned} dw &= -\frac{\mathbf{x} \cdot d\mathbf{x}}{\lambda w} \\ dw^2 &= \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{w^2} \\ &= \frac{\lambda (\mathbf{x} \cdot d\mathbf{x})^2}{\lambda R^2 - \mathbf{x} \cdot \mathbf{x}} \end{aligned}$$

so the line element ds^2, restricted to the spherical surface, is (with $\lambda^2 = 1$)

$$ds^2 = dx^2 + dy^2 + dz^2 + \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{\lambda R^2 - \mathbf{x} \cdot \mathbf{x}}$$

Let $\kappa \equiv \frac{\lambda}{R^2}$ and $\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^2$ to write this as

$$\begin{aligned} ds^2 &= d\mathbf{x}^2 + \frac{\kappa (\mathbf{x} \cdot d\mathbf{x})^2}{1 - \kappa \mathbf{x}^2} \\ &= h_{ij} dx^i dx^j \end{aligned}$$

This gives us the metric

$$h_{ij} = \delta_{ij} + \frac{\kappa x_i x_j}{1 - \kappa \mathbf{x}^2}$$

Show directly by computing the curvature tensor that this describes a 3-dimensional space of constant curvature.

3. Show that the geodesic equation for the radial component of the 4-velocity,

$$\frac{du^1}{d\tau} = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right) u^0 u^0 + \frac{M}{r^2\left(1 - \frac{2M}{r}\right)} u^1 u^1 + r\left(1 - \frac{2M}{r}\right) u^2 u^2 + \left(1 - \frac{2M}{r}\right) r \sin^2\theta u^3 u^3$$

follows from the other two geodesic equations

$$\begin{aligned} \frac{dt}{d\tau} &= u_0^0 \left(\frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}}\right) \\ \frac{d\varphi}{d\tau} &= \frac{L}{r^2} \end{aligned}$$

and the norm of u^α,

$$\left.\begin{matrix} \pm \\ 0 \end{matrix}\right\} = \left(1 - \frac{2M}{r}\right)\left(u^0\right)^2 + \frac{1}{1 - \frac{2M}{r}}\left(u^1\right)^2 + r^2\left(u^3\right)^2$$

by differentiating with respect to τ (or λ for the null case) and substituting for $u^0 = \frac{dt}{d\tau}$ and $u^3 = \frac{d\varphi}{d\tau}$.