Manifolds, vectors and forms

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1 Manifolds

1.1 Intuitions

Loosely speaking, a manifold is a (topological) space that looks like a small piece of R^n in any sufficiently small region. For example, the 2-dimensional surface of a ball in 3-dimensions it the space S^2. If we move very close to the surface, it looks like a piece of a Euclidean plane. Indeed, the distance between two nearby points on the surface of a sphere of radius R is

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Pick a point, $x^i = (\theta_0, \varphi_0)$, and look in a nearby region of the surface. Expanding about that point, in a region $(\theta_0 + \varepsilon, \varphi_0 + \sigma)$,

$$\begin{aligned} \sin(\theta_0 + \varepsilon) &= \sin\theta_0 + \cos\theta_0\varepsilon + \ldots \\ d\theta &= d\varepsilon \\ d\varphi &= d\sigma \end{aligned}$$

the distance becomes

$$\begin{aligned} ds^2 &= R^2 \left(d\varepsilon^2 + (\sin\theta_0 + \cos\theta_0\varepsilon + \ldots)^2 d\sigma^2\right) \\ &\approx R^2 \left(d\varepsilon^2 + \sin^2\theta_0 d\sigma^2\right) \end{aligned}$$

Now define new coordinates

$$\begin{aligned} x &= R\varepsilon \\ y &= R\sigma\sin\theta_0 \end{aligned}$$

and as long as we can ignore the terms of order $\varepsilon\,(d\sigma)^2$, we have

$$ds^2 \approx dx^2 + dy^2$$

The sphere looks like a plane when we get close enough. Notice that at every point of S^2 the sphere looks like a *plane* – it is always two dimensional.

Now picture the 2-dimensional surface of an egg. It is similar to a sphere, but stretched out along one axis. Any small bit also looks like a small piece of a plane, and is always 2-dimensional. This shape is also a manifold; in fact, it *also* S^2. The property of being a manifold does not have rigidity. We may smoothly deform our picture of a manifold, and it remains the same manifold. What makes a difference is punching holes or joining boundaries. Thus, for example, the plane R^2 is a manifold and so is any wavy deformation of the plane. R^2 and S^2 are distinct manifolds because we cannot smoothly deform one into the other without making a puncture in the sphere or sewing up the "edges" of the plane.

A manifold does *not* necessarily have a metric. Though we used the metric of a 2-sphere above to introduce the idea of a manifold being locally Euclidean, it is the metric that fixes the difference between an egg and a sphere. Our definition below does not rely on having a metric. In general relativity we will be interested in manifolds with metric, $(\mathcal{M}, g)$.

A third example of a 2-dimensional manifold is the toroidal surface of a donut. This is distinct from both R^2 and S^2. If we run a circle around the outer limit of the torus (labeled by $0 \leq \theta < 2\pi$), and a perpendicular circle around the tube of the torus (labeled by $0 \leq \varphi < 2\pi$), we see that we can label every point of the surface by a pair of angles (θ, φ). The surface is the direct product of these two circles, $S^1 \otimes S^1$.

Higher dimensional manifolds are often difficult to picture, but many may be classified.

1.2 Abstractions

The key to identifying a manifold lies in the property of looking like flat, n-dimensional space, R^n, in a small enough region around each point. We state this property in terms of a $1-1$, onto mapping called a *chart*.

Concretely, in order for a space to be an n-dimensional manifold, $\mathcal{M}^n$, we start with the requirement that in a neighborhood $N(\mathcal{P})$ of each point $\mathcal{P}$, there must be a $1-1$, onto mapping, ϕ, to an open subset, $O(\mathbf{x}_0)$, in R^n, where we take $\phi(\mathcal{P}) = \mathbf{x}_0$

$$\phi : N(\mathcal{P}) \leftrightarrow O(\mathbf{x}_0)$$

For any point in the open set $\mathcal{Q} \in N(\mathcal{P})$, there is a unique $\mathbf{x} \in O(\mathbf{x}_0)$ with

$$\phi(\mathcal{Q}) = \mathbf{x}$$

In R^n, $\mathbf{x}$ is an n-tuple of numbers and these are the coordinates of the point $\mathcal{Q}$.

We cannot usually find a single such a mapping ϕ that assigns coordinates to every point $\mathcal{M}^n$, so we need to specify how the coordinates in nearby regions are related. Let $\mathcal{P}_1, \mathcal{P}_2$ be points in $\mathcal{M}^n$ with neighborhoods $N_1(\mathcal{P}_1)$ and $N_2(\mathcal{P}_2)$ and charts ϕ_1 and ϕ_2. Then for all points in the intersection

$$N_{12} = N_1(\mathcal{P}_1) \cap N_2(\mathcal{P}_2)$$

we have two different sets of coordinates, $O_1 = \phi_1(N_{12})$ and $O_2 = \phi_2(N_{12})$. We require there to exist a sensible transformation between these.

The relationship must hold between the coordinates $\phi_1(N_{12})$ and $\phi_2(N_{12})$, and we can specify the relationship by using the inverse mapping, ϕ_1^{-1}. We apply two maps in succession. The first,

$$\phi_1^{-1} : \quad O_1 \quad \rightarrow \quad N_{12}$$

maps from the open region O_1 in R^n to the overlap region N_{12}. From here we map again with ϕ_2,

$$\phi_2 : N_{12} \rightarrow O_2$$

taking points of the overlap to the open region O_2 in R^n. The combination of these, $\phi_2 \circ \phi_1^{-1}$ is a map between two open sets in R^n,

$$\phi_2 \circ \phi_1^{-1} : \quad O_1 \quad \rightarrow \quad O_2$$

so that the composition of the two maps, maps one open region of R^n to another,

$$\phi_2 \circ \phi_1^{-1} : O_1 \rightarrow O_2$$

We require the mapping $\phi_2 \circ \phi_1^{-1}$ from $O_1 \subset R^n$ to $O_2 \subset R^n$ to be infinitely (n-times) differentiable. $\mathcal{M}^n$ is then a C^∞ (C^n)manifold.

1.3 Examples

1.3.1 The plane, R^2

The mapping of points of the plane to Cartestian coordinates, (x, y), is $1-1$ and onto the entire plane. We require only one chart, although we may choose countless others. For example, consider the case of polar coordinates, (ρ, φ). These cover any open set that excludes the origin, but at the origin $(0, \varphi) \leftrightarrow (0, 0)$ is no longer single valued. We require a second chart. Choose a small open rectangle including the origin, and let the coordinates there be Cartesian. The overlap region is the rectangle minus the point at the origin. In this region the two charts are related by

$$\begin{aligned} x &= \rho \cos \varphi \\ y &= \rho \sin \varphi \\ \rho &= \sqrt{x^2 + y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

These relations are smooth functions, and at every point we require exactly two coordinates, so we have a 2-dimensional, C^∞ manifold.

1.3.2 The 2-sphere, S^2

A natural choice for ϕ is the usual assignment of (θ, ϕ) but again these are not single valued at the poles. In fact, S^2 *requires* at least two charts, and this proves that S^2 is a different manifold from R^2. Two charts is sufficient, however. For the first, begin with polar coordinates centered at the north pole but, to include the north pole, transform to Cartesian,

$$\begin{aligned} x &= \rho \cos \varphi \\ y &= \rho \sin \varphi \end{aligned}$$

Then the north pole has the unique coordinates $(x, y) = (0, 0)$. This map remains single valued all the way to, but not including, the south pole where $(x, y) = (\pi R \cos \varphi, \pi R \sin \varphi)$. Unlike the north pole, different values of φ now give different values for x and y, although the point remains the same. We require another chart. For the second chart, we again choose polar coordinates (r, θ), where r is measured from the south pole and θ increases counterclockwise when we look down on the south pole. Converting to Cartesian coordinates (u, v), we set

$$\begin{aligned} u &= r \cos \theta \\ v &= r \sin \theta \end{aligned}$$

This is single valued everywhere except the north pole.

Now, we look in the overlap region, consisting of the entire sphere minus the two poles. The mapping between (ρ, φ) and (r, θ) is given by

$$\begin{aligned} r &= \pi R - \rho \\ \theta &= -\varphi \end{aligned}$$

so that in this region

$$\begin{aligned} u &= (\pi R - \rho) \cos \varphi \\ &= \pi R \cos \varphi - \rho \cos \varphi \\ &= \frac{\pi R x}{\sqrt{x^2 + y^2}} - x \end{aligned}$$

$$\begin{aligned} v &= -(\pi R - \rho)\sin\varphi \\ &= -\pi R\sin\varphi + \rho\sin\varphi \\ &= -\frac{\pi R y}{\sqrt{x^2+y^2}} + y \end{aligned}$$

and these are infinitely differentiable in the overlap region. Again, we require exactly two coordinates at each point of S^2, so we have a 2-dimensional, C^∞ manifold. Since we cannot cover the space with one chart, the manifold is different from the plane.

Exercise: Circle

Prove that a circle is a manifold by specifying an appropriate chart or charts. What is the minimum number of charts required?

Exercise: Torus

A simple way to define a torus is to put periodic boundary conditions on a rectangle. For concreteness, consider the rectangle

$$\begin{aligned} x &\in [0,a] \\ y &\in [0,b] \end{aligned}$$

and identify points along the top and bottom boundaries,

$$(x,0) \equiv (x,b)$$

and along the left and right boundaries,

$$(0,y) \equiv (a,y)$$

It is easy to assign charts in Cartesian coordinates. Find a set of charts to show that the torus is a manifold, and find the minimum number of charts required to prove that the torus is a distinct manifold from R^2 or S^2.

2 Vectors and forms

We now define two vector spaces associated with any manifold. Both spaces depend on two simple ideas: functions and curves.

2.1 Functions and curves

A real-valued function on a manifold is an assignment of a real number to each point of the manifold,

$$f : \mathcal{M} \to R$$

By using the charts of the manifold, we can differentiate the function. For any point $\mathcal{P}$ of $\mathcal{M}$, there exists a chart on some neighborhood, $N(\mathcal{P})$, of $\mathcal{P}$,

$$\phi : N(\mathcal{P}) \leftrightarrow O(\mathbf{x}_0)$$

so combining with the function for each point in $N(\mathcal{P})$ we have a mapping from a region in R^n to the reals,

$$f \circ \phi^{-1} : O(\mathbf{x}_0) \to R$$

We may write the result of this as map as the number $f(\mathbf{x}) \in R$, where $\mathbf{x}$ is a point in R^n. Then f is a real-valued function on R^n and we may differentiate it in the usual way,

$$\frac{\partial f}{\partial x^\alpha}$$

While functions map from $\mathcal{M}$ to R, a curve is a mapping from R into $\mathcal{M}$:

$$C : R \to \mathcal{M}$$

Combined with a chart

$$\phi^{-1} \circ C : \ R \to O(\mathbf{x}_0)$$

we have a parameterized curve in R^n, $\mathbf{x}(\lambda)$, where as $\lambda \in R$ varies, the point $\mathbf{x}(\lambda)$ traces out a path in R^n.

2.2 Vectors

We define:

Def: A vector at a point $\mathcal{P}$ is a directional derivative at $\mathcal{P}$ along a curve C Consider the values of a function $f(\mathcal{P})$ restricted to a curve $C(\lambda)$, $f(C(\lambda)) = f \circ C : R \to R$. The derivative

$$\frac{df}{d\lambda}$$

tells us how the function f is changing along the curve C. This is intrinsic to the space. The function at any point of the curve C is a number, and our usual definition of derivative works:

$$\frac{df}{d\lambda} = \lim_{\varepsilon \to 0} \frac{f(C(\lambda+\varepsilon)) - f(C(\lambda))}{\varepsilon}$$

Here, λ and ε are real numbers, $C(\lambda)$ and $C(\lambda+\varepsilon)$ are points of the manifold, and $f(C(\lambda))$ is another number, the value of the function f at the point $C(\lambda)$.

Using a chart, we may write

$$\begin{aligned} f(\mathcal{P}) &= f \circ \phi^{-1} \circ \phi(\mathcal{P}) \\ &= \left(f \circ \phi^{-1}\right)(\phi(\mathcal{P})) \\ &= F(x^\alpha) \end{aligned}$$

Here, $f \circ \phi^{-1}$ maps a point with coordinates x^α in R^n to a point $\mathcal{P}$ of the manifold, then f takes a value at that point. Then, to evaluate the derivative along C, where

$$\phi(C(\lambda)) = x^\alpha(\lambda)$$

we have

$$\begin{aligned} \frac{df}{d\lambda} &= \lim_{\varepsilon \to 0} \frac{f(C(\lambda+\varepsilon)) - f(C(\lambda))}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{f \circ \phi^{-1} \circ \phi(C(\lambda+\varepsilon)) - f \circ \phi^{-1} \circ \phi(C(\lambda))}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{f(x^\alpha(\lambda+\varepsilon)) - f(x^\alpha(\lambda))}{\varepsilon} \\ &= \frac{df(x^\alpha(\lambda))}{d\lambda} \\ &= \frac{\partial f(x^\alpha)}{\partial x^\alpha} \frac{dx^\alpha(\lambda)}{d\lambda} \end{aligned}$$

Therefore, in coordinates, i.e., a basis, we may write

$$\frac{d}{d\lambda} f(\lambda) = \frac{dx^\alpha}{d\lambda} \frac{\partial}{\partial x^\alpha} f(x^\alpha)$$

We now define a *vector* to be the directional derivative operator

$$\frac{d}{d\lambda} = \frac{dx^\alpha}{d\lambda} \frac{\partial}{\partial x^\alpha}$$

Which function we act on is irrelevant. We see that once we make a choice of coordinates x^α, all directional derivatives may be written as linear combinations of the *basis vectors*

$$\vec{e}_\alpha = \frac{\partial}{\partial x^\alpha}$$

The coefficients of this linear combination are the tangents to the mapped curve, $x^\alpha(\lambda) = \phi \circ C(\lambda)$,

$$v^\alpha = \frac{dx^\alpha}{d\lambda}$$

We now show that these directional derivatives form a vector space. It is not hard to see that scalar multiples are also curves, since a change of parameter from λ to $a\lambda$ changes v^α to av^α so that av^α is also a vector. The only tricky part of the demonstration is to show that we can add directional derivatives to get a third directional derivative. We content ourselves with demonstrating this.

Suppose we have two curves, $C_1(\lambda)$ and $C_2(\lambda)$, both passing through the same point P. Then

$$\begin{aligned} \phi \circ C_1 &= x_1^\alpha(\lambda) \\ \phi \circ C_2 &= x_2^\alpha(\lambda) \end{aligned}$$

where $x_1^\alpha(\lambda), x_2^\alpha(\lambda)$ are two curves in R^n. We may choose the mappings C_1 and C_2 so that $C_1(0) = C_2(0) = P$. Since R^n is a vector space, we may add the vectors x_1^α and x_2^α at each value of λ to get a new curve

$$x_3^\alpha(\lambda) = (x_1^\alpha + x_2^\alpha)(\lambda)$$

Then

$$\begin{aligned} C_3(\lambda) &= \phi^{-1}(x_3^\alpha(\lambda)) \\ &= \phi^{-1}(\phi \circ C_1 + \phi \circ C_2) \end{aligned}$$

is a curve in $\mathcal{M}$. Since $\phi \circ C_3$ is just $x_3^\alpha(\lambda)$, the directional derivative along C_3 is

$$\begin{aligned} \frac{d}{d\lambda_{(3)}} &= \frac{dx_3^\alpha(\lambda)}{d\lambda} \frac{\partial}{\partial x^\alpha} \\ &= \frac{d(x_1^\alpha + x_2^\alpha)(\lambda)}{d\lambda} \frac{\partial}{\partial x^\alpha} \\ &= \frac{dx_1^\alpha}{d\lambda} \frac{\partial}{\partial x^\alpha} + \frac{dx_2^\alpha}{d\lambda} \frac{\partial}{\partial x^\alpha} \\ &= \frac{d}{d\lambda_{(1)}} + \frac{d}{d\lambda_{(2)}} \end{aligned}$$

The sum of two directional derivatives is therefore a third directional derivative. This, together with the usual properties of addition and scalar multiplication, show that directional derivatives form a vector space.

Exercise:

Define real linear combinations of directional derivatives, $\alpha \frac{d}{d\lambda}_{(1)} + \beta \frac{d}{d\lambda}_{(2)}$ in such a way that the distributive laws,

$$
\begin{aligned}
(\alpha+\beta) \frac{d}{d\lambda} &= \alpha \frac{d}{d\lambda} + \beta \frac{d}{d\lambda} \\
\alpha \left(\frac{d}{d\lambda}_{(1)} + \frac{d}{d\lambda}_{(2)} \right) &= \alpha \frac{d}{d\lambda}_{(1)} + \alpha \frac{d}{d\lambda}_{(2)}
\end{aligned}
$$

are satisfied and prove that $\alpha \frac{d}{d\lambda}_{(1)} + \beta \frac{d}{d\lambda}_{(2)}$ is a directional derivative along some curve C_3.

2.3 Forms

There is a second vector space arising from curves and functions on a manifold.

Def: A form is a linear map on curves The basic idea here is that an integral is a linear mapping. If we integrate the differential of a function along a curve, we get a number,

$$
f(x^\alpha) = \int_C^{x^\alpha} df
$$

The differentials, df, combine linearly,

$$
a\, f(x^\alpha) + b\, g(x^\alpha) = \int_C^{x^\alpha} (a\, df + b\, dg)
$$

Therefore, we may regard df, dg as linear mappings that take the curve C into the reals, R. The linear combination $a\, df + b\, dg$ is another such mapping. What we need to do is define these things in a way that applies to general manifolds.

Consider an arbitrary linear mapping on curves,

$$
\tilde{\omega} : C \to R
$$

Linearity guarantees that for any such mapping, we can divide the curve $C(\lambda)$ into small pieces,

$$
C_k(\lambda) = \{ C(\lambda) \mid \lambda \in [\lambda_k, \lambda_{k+1}] \}
$$

so that C_k is the piece of C running from parameter values λ_k to λ_{k+1}. Clearly,

$$
C(\lambda) = \sum_{k=0}^{n} C_k(\lambda)
$$

and by the linearity of $\tilde{\omega}$,

$$
\tilde{\omega}(C(\lambda)) = \sum_{k=0}^{n} \tilde{\omega}(C_k(\lambda))
$$

Now use charts to write this in coordinates:

$$
\tilde{\omega}(C(\lambda)) = \sum_{k=0}^{n} \tilde{\omega} \circ \phi^{-1} \circ \phi \circ C_k(\lambda)
$$

where $\phi \circ C_k(\lambda) : R \rightarrow R^n$ is a curve, $x^\alpha(\lambda)$, in R^n. Then $\tilde{\omega} \circ \phi^{-1}$ acts on this curve, mapping first to $C(x^\alpha(\lambda))$ then to the value $f(\lambda) = \tilde{\omega}(C(x^\alpha(\lambda)))$. This makes the right side a mapping from points along a curve, $x^\alpha(\lambda)$, in R^n to the reals, R, giving a function, $f(\lambda)$.

Now consider the sum,

$$\sum_{k=0}^{n} \tilde{\omega} \circ \phi^{-1} \circ \phi \circ C_k(\lambda)$$

Let n become large so that $\lambda_{k+1} - \lambda_k \rightarrow d\lambda$. Then $\phi \circ C_k(\lambda)$ is just the coordinate change, $dx^\alpha(\lambda) = \frac{dx^\alpha}{d\lambda} d\lambda$, for an infinitesimal piece of the curve from λ to $\lambda + d\lambda$. The form returns the value a real number which must depend linearly on this coordinate displacement,

$$\tilde{\omega} \circ \phi^{-1}(dx^\alpha) = \omega_\alpha dx^\alpha$$

In the limit, the sum becomes an integral along the curve C, $\lim_{d\lambda \rightarrow 0} \sum_{k=0}^{n} \omega_\alpha dx^\alpha = \int_C \omega_\alpha dx^\alpha$, so that

$$\tilde{\omega}(C(\lambda)) = \int_C \omega_\alpha dx^\alpha$$

Again, we make an operator interpretation. The form is the integrand,

$$\tilde{\omega} = \omega_\alpha dx^\alpha$$

The components of the form are ω_α and the coordinate differentials dx^α form a basis. The operation of $\tilde{\omega}$ on any curve C is the integral of $\tilde{\omega}$ along the curve,

$$\tilde{\omega} : C \rightarrow R$$

where $\tilde{\omega}(C) \equiv \int_C \omega_\alpha dx^\alpha$. Just as for our definition of a directional derivative we ignored which function we differentiate, we now neglect which curve we integrate along and focus on the 1-*form*,

$$\tilde{\omega} = \omega_\alpha dx^\alpha$$

These form a vector space, with components ω_α and basis forms, dx^α.

Mappings between vectors and forms

Writing the integral $\tilde{\omega}(C) = \int_C \omega_\alpha dx^\alpha$ as an integral over the parameter λ,

$$\begin{aligned} \tilde{\omega}(C(\lambda)) &= \int_C \omega_\alpha \frac{dx^\alpha}{d\lambda} d\lambda \\ &= \int_C f(\lambda) \, d\lambda \end{aligned}$$

the integrand shows us a mapping to the reals, given by combining the form with a tangent vector, $\omega_\alpha \frac{dx^\alpha}{d\lambda} = f(\lambda)$. This is just the mapping of $\tilde{\omega}$ on an infinitesmal curve $\frac{dx^\alpha}{d\lambda} d\lambda$. Now, writing the tangent vector in components, $\phi(\vec{t}) = \frac{dx^\alpha}{d\lambda}$, we may also write the form as a linear mapping on vectors,

$$\tilde{\omega}(\vec{t}) = \omega_\alpha \frac{dx^\alpha}{d\lambda}$$

This expression, like $\tilde{\omega}(C)$, is independent of the basis.