The standard model: Lambda-CDM

April 17, 2015

Currently, the simplest cosmological model that fits the experimental observations is the Λ-CDM model. Lambda is the cosmological constant and CDM stands for cold dark matter. Our Friedmann equation requires only minor modification to describe Λ-CDM – the inclusion of radiation. As the universe expanded, it cooled, but before about 380,000 years after the big bang, it was too hot for eletrons and

1 Ultrarelativistic energy momentum tensor

Going back in time toward the big bang, when matter becomes ultrarelativistic, we need a different energy tensor because the contribution of pressure becomes important. In the extreme case of pure radiation, we have $\rho = 3p$. Maintaining homogeneity and isotropy, we may only use the metric and the comoving velocity:

$$\begin{aligned} T^{\alpha\beta} &= (\rho + p) u^\alpha u^\beta + p g^{\alpha\beta} \\ &= \begin{pmatrix} \rho & \\ & p g^{ij} \end{pmatrix} \end{aligned}$$

where, in the comoving frame, $u^a = (1,0,0,0)$ and $g^{ij} = \frac{1}{a^2} h^{ij}$. The conservation equation requires,

$$\begin{aligned} 0 &= T^{\mu\nu}{}_{;\nu} \\ &= T^{\mu\nu}{}_{,\nu} + T^{\beta\nu}\Gamma^{\mu}_{\beta\nu} + T^{\mu\beta}\Gamma^{\nu}_{\beta\nu} \end{aligned}$$

Recalling the form of the connection,

$$\begin{aligned} \Gamma^i_{0j} &= \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^0_{ij} &= a\dot{a}h_{ij} \\ \Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} \end{aligned}$$

the time component is

$$\begin{aligned} 0 &= T^{0\nu}{}_{,\nu} + T^{\beta\nu}\Gamma^0_{\beta\nu} + T^{0\beta}\Gamma^\nu_{\beta\nu} \\ &= T^{00}{}_{,0} + p g^{ij}\Gamma^0_{ij} + T^{00}\Gamma^\nu_{0\nu} \\ &= \dot{\rho} + p g^{ij} a\dot{a}h_{ij} + \frac{3\dot{a}}{a}\rho \\ &= \dot{\rho} + \frac{3\dot{a}}{a}p + \frac{3\dot{a}}{a}\rho \\ &= \dot{\rho} + \frac{4\dot{a}}{a}\rho \end{aligned}$$

Multiplying by a^4 we have

$$\begin{aligned} 0 &= a^4\dot{\rho} + 4a^3\dot{a}\rho \\ &= \frac{d}{dt}\left(\rho a^4\right) \end{aligned}$$

so that ρa^4 remains constant.

$$\rho a^4 = \varepsilon = constant$$

This means that $\rho = \frac{\alpha}{a^4}$ is a function of time only, so the pressure is too. Therefore, for the spatial components,

$$\begin{aligned} 0 &= T^{i\nu}{}_{,\nu} + T^{\beta\nu}\Gamma^{i}{}_{\beta\nu} + T^{i\beta}\Gamma^{\nu}{}_{\beta\nu} \\ &= T^{ij}{}_{,j} + T^{jk}\Gamma^{i}{}_{jk} + T^{ij}\Gamma^{k}{}_{jk} \\ &= \left(pg^{ij}\right)_{,j} + pg^{jk}\tilde{\Gamma}^{i}_{jk} + pg^{ij}\tilde{\Gamma}^{k}{}_{jk} \\ &= \frac{p}{a^2}\left(h^{ij}{}_{,j} + h^{jk}\tilde{\Gamma}^{i}_{jk} + h^{ij}\tilde{\Gamma}^{k}{}_{jk}\right) \\ &\equiv 0 \end{aligned}$$

since $h^{ij}{}_{,j} + h^{jk}\tilde{\Gamma}^{i}_{jk} + h^{ij}\tilde{\Gamma}^{k}{}_{jk}$ is the covariant derivative of the constant curvature metric on the constant curvature submanifold.

2 Radiation cosmology

Suppose we have a universe filled with radiation. In the early universe, after inflation but before the current matter dominated era, this is a good approximation.

The field equations are now

$$\begin{aligned} G_{00} + \Lambda g_{00} &= \beta T_{00} \\ \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) - \Lambda &= \frac{\beta\varepsilon}{a^4} \end{aligned}$$

while the spatial components give a second equation,

$$\begin{aligned} G_{ij} + \Lambda g_{ij} &= \beta T_{ij} \\ G_{ij} + \Lambda g_{ij} &= \beta p a^2 h_{ij} \\ -\left(2a\ddot{a} + \kappa + \dot{a}^2 - \Lambda a^2\right) h_{ij} &= \frac{1}{3}\beta\rho a^2 h_{ij} \\ -\left(2a\ddot{a} + \kappa + \dot{a}^2 - \Lambda a^2\right) &= \frac{1}{3}\beta\frac{\varepsilon}{a^2} \end{aligned}$$

and therefore,

$$\begin{aligned} 3a^2\left(\kappa + \dot{a}^2\right) - \Lambda a^4 - \beta\varepsilon &= 0 \\ 3a^2\left(2a\ddot{a} + \kappa + \dot{a}^2\right) - 3\Lambda a^4 + \beta\varepsilon &= 0 \end{aligned}$$

Multiply the first by a and differentiate

$$\begin{aligned} 3a^3\left(\kappa + \dot{a}^2\right) - \Lambda a^5 - \beta\varepsilon a &= 0 \\ 9a^2\kappa\dot{a} + 9a^2\dot{a}^3 + 6a^3\dot{a}\ddot{a} - 5\Lambda a^4\dot{a} - \beta\varepsilon\dot{a} &= 0 \\ 3a^2\left(2a\ddot{a} + 3\left(\kappa + \dot{a}^2\right)\right) - 5\Lambda a^4 - \beta\varepsilon &= 0 \end{aligned}$$

Subtract,

$$\begin{aligned} 6a^2\left(\kappa + \dot{a}^2\right) - 2\Lambda a^4 - 2\beta\varepsilon &= 0 \\ 3a^2\left(\kappa + \dot{a}^2\right) - \Lambda a^4 - \beta\varepsilon &= 0 \end{aligned}$$

and this is satisfied by the first equation, so once again we need only the Friedmann equation.

Now look at the first equation,

$$-\kappa = \dot{a}^2 - \frac{\Lambda a^4}{3a^2} - \frac{\beta\varepsilon}{3a^2}$$

This is essentially the same as before, with the effective potential

$$V_{eff} = -\frac{1}{3}\Lambda a^2 - \frac{\beta\varepsilon}{3a^2}$$

except that the divergence of the potential as $a \to 0$ goes as $V_{eff} \sim \frac{1}{a^2}$.

3 A realistic cosmology: the Λ-CDM model

To build a realistic cosmological model, we divide the energy-momentum sources into four contributions:

1. Normal massive particles, $\rho_m = \frac{m}{a^3}$
2. Radiation, $\rho_r = \frac{\varepsilon}{a^4}$
3. Dark matter, $\rho_d = \frac{m_d}{a^3}$
4. Cosmological constant, Λ

The dark matter behaves in much the same way as normal matter, but it is useful to separate it out for experimental purposes. Sometimes the cosmological constant is absorbed into the stress-energy tensor and treated as a field rather than a constant. Though there is no evidence for such a field, called *quintessence*, it motivates experimental searches.

Putting together all components, we have the single Einstein equation

$$\frac{\dot{a}^2}{a^2} = \frac{\beta\varepsilon}{3a^4} + \frac{\beta m}{3a^3} + \frac{\beta m_d}{3a^3} - \frac{\kappa}{a^2} + \frac{1}{3}\Lambda$$

Generally, instead of choosing units so that $\kappa = \pm 1, 0$, the units are chosen so that $a(t)$ at the present time is 1. This sets the scale for the other quantities. Also, some define $\Omega_r = \frac{\beta\varepsilon}{3}, \Omega_m = \frac{\beta m}{3}, \Omega_\kappa = -\kappa$, and $\Omega_\Lambda = \frac{1}{3}\Lambda$.

Increasingly accurate measurements show that within experimental error, $\kappa = 0$, so we consider the effective potential for this case.

Letting $M = m + m_d$, we write,

$$-\kappa = 0 = \dot{a}^2 - \frac{\beta\varepsilon}{3a^2} - \frac{\beta M}{3a} - \frac{1}{3}\Lambda a^2$$

so the effective potential is

$$V_{eff} = -\frac{\beta\varepsilon}{3a^2} - \frac{\beta M}{3a} - \frac{1}{3}\Lambda a^2$$

with extrema when

$$\begin{aligned} 0 &= \frac{dV_{eff}}{da} \\ &= \frac{2\beta\varepsilon}{3a^3} + \frac{\beta M}{3a^2} - \frac{2}{3}\Lambda a \\ 0 &= \frac{2\beta\varepsilon}{3} + \frac{\beta M}{3}a - \frac{2}{3}\Lambda a^4 \end{aligned}$$

For $\Lambda > 0$ (in agreement with experiment),

$$a_{max}^4 = \frac{\beta M}{2\Lambda}\left(a_{max} + \frac{2\varepsilon}{m}\right)$$

The right side grows faster near $a = 0$, so the curves will cross at exactly one point. Therefore, there is a single extremum. The second derivative,

$$\frac{d^2 V_{eff}}{da^2} = -\frac{2\beta\varepsilon}{a^4} - \frac{2\beta M}{3a^3} - \frac{2}{3}\Lambda < 0$$

so the single extremum is a maximum. Finally, we check the value of V_{eff} at the maximum,

$$\begin{aligned}
V_{eff}(a_{max}) &= \frac{\beta}{3a^2}\left(-\varepsilon - Ma_{max} - \frac{1}{\beta}\Lambda a_{max}^4\right) \\
&= \frac{\beta}{3a^2}\left(-\varepsilon - Ma_{max} - \frac{1}{\beta}\Lambda\frac{\beta M}{2\Lambda}\left(a_{max} + \frac{2\varepsilon}{m}\right)\right) \\
&= \frac{\beta}{3a^2}\left(-\varepsilon - Ma_{max} - \frac{M}{2}a_{max} - \varepsilon\right) \\
&= \frac{\beta}{3a^2}\left(-\frac{3M}{2}a_{max} - 2\varepsilon\right)
\end{aligned}$$

Since this is less than zero, while the effective energy is indistinguishable from zero, $-\kappa \approx 0$, the universe has enough energy to escape the initial singularity and expand forever.

4 Inflation

At early times when a is small, all the matter is ultrarelativistic and the effective potential is dominated by the first term,

$$\begin{aligned}
\frac{\dot{a}^2}{a^2} &= \frac{\beta\varepsilon}{3a^4} \\
a\dot{a} &= +\sqrt{\frac{\beta\varepsilon}{3}} \\
\frac{1}{2}a^2 &= +\sqrt{\frac{\beta\varepsilon}{3}}t \\
a &= \left(\frac{4\beta\varepsilon}{3}\right)^{1/4}\sqrt{t}
\end{aligned}$$

which is a fairly slow rate of expansion. There is a serious problem with this, and there are too many solutions.

4.1 The horizon problem

When we look a great distance in any direction, we see the same temperature for the cosmic microwave background. However, with such a slow rate of expansion as given above, these regions have always been at a great distance from one another – they were never close enough to achieve such a uniform temperature. This problem is solved (though others may arise) if during the earliest epoch the universe expanded exponentially until it increased in size by a factor of at least 10^{26}.