A worked example: Geodesics on a parabolic surface

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1 The metric

Consider the two dimensional surface in 3-space described in polar coordinates by an axisymmetric shape with parabolic cross section,

$$z = \frac{1}{2}a\rho^2$$

The metric is the metric of Euclidean 3-space, reduced by this parabolic constraint, $(\rho, \varphi, z) = \left(\rho, \varphi, \frac{1}{2}a\rho^2\right)$. Taking the differential of z, we have $dz = a\rho d\rho$. Therefore,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2 d\varphi^2 + dz^2 \\ &= d\rho^2 + \rho^2 d\varphi^2 + a^2\rho^2 d\rho^2 \\ &= \left(1 + a^2\rho^2\right) d\rho^2 + \rho^2 d\varphi^2 \end{aligned}$$

and we read of the metric,

$$g_{ij} = \begin{pmatrix} 1 + a^2\rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

with inverse

$$g^{ij} = \begin{pmatrix} \frac{1}{1+a^2\rho^2} & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}$$

2 The connection

There are two nonvanishing derivatives of the metric,

$$\begin{aligned} g_{11,1} &= 2a^2\rho \\ g_{22,1} &= 2\rho \end{aligned}$$

This means that the only nonvanishing connection coefficients are $\Gamma_{111}, \Gamma_{221} = \Gamma_{212}, \Gamma_{122}$, given by:

$$\begin{aligned} \Gamma_{111} &= \frac{1}{2}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) \\ &= a^2\rho \\ \Gamma_{221} = \Gamma_{212} &= \frac{1}{2}\left(g_{22,1} + g_{21,2} - g_{12,2}\right) \\ &= \frac{1}{2}\left(2\rho\right) \\ &= \rho \\ \Gamma_{122} &= \frac{1}{2}\left(g_{12,2} + g_{12,2} - g_{22,1}\right) \\ &= -\rho \end{aligned}$$

Now, raising the first index on each,

$$\begin{aligned}\Gamma^1_{\ ij} &= g^{1k}\Gamma_{kij} \\ &= g^{11}\Gamma_{1ij}\end{aligned}$$

There are two cases,

$$\begin{aligned}\Gamma^1_{\ 11} &= \frac{a^2\rho}{1+a^2\rho^2} \\ \Gamma^1_{\ 22} &= -\frac{\rho}{1+a^2\rho^2}\end{aligned}$$

Then for the $\Gamma^2_{\ ij}$,

$$\begin{aligned}\Gamma^2_{\ ij} &= g^{2k}\Gamma_{kij} \\ &= g^{22}\Gamma_{2ij} \\ \Gamma^2_{\ 21} &= g^{22}\Gamma_{221} \\ &= \frac{1}{\rho}\end{aligned}$$

3 The geodesic equation

The geodesic equation is

$$\begin{aligned}0 &= v^i D_i v^j \\ &= v^i\left(\partial_i v^j + v^k\Gamma^j_{\ ki}\right)\end{aligned}$$

where the tangent to the geodesic is $v^i = \frac{dx^i}{dx}$.

For $j = 1$,

$$\begin{aligned}0 &= v^i\partial_i v^1 + v^i v^k\Gamma^1_{\ ki} \\ &= \frac{dv^1}{ds} + v^1v^1\Gamma^1_{\ 11} + v^2v^2\Gamma^1_{\ 22} \\ &= \frac{dv^1}{ds} + \frac{a^2\rho}{1+a^2\rho^2}\left(v^1\right)^2 - \frac{\rho}{1+a^2\rho^2}\left(v^2\right)^2\end{aligned}$$

For $j = 2$,

$$\begin{aligned}0 &= v^i\partial_i v^2 + v^i v^k\Gamma^2_{\ ki} \\ &= \frac{dv^2}{ds} + v^2v^1\Gamma^2_{\ 21} + v^1v^2\Gamma^2_{\ 12} \\ &= \frac{dv^2}{ds} + \frac{2}{\rho}v^1v^2\end{aligned}$$

We therefore have a pair of equations,

$$\begin{aligned}0 &= \frac{dv^1}{ds} + \frac{a^2\rho}{1+a^2\rho^2}\left(v^1\right)^2 - \frac{\rho}{1+a^2\rho^2}\left(v^2\right)^2 \\ 0 &= \frac{dv^2}{ds} + \frac{2}{\rho}v^1v^2\end{aligned}$$

4 Integrating the geodesic equation

It is helpful to notice that the components of the vector v^i are related by the line element,

$$ds^2 = \left(1 + a^2\rho^2\right) d\rho^2 + \rho^2 d\varphi^2$$

Dividing by ds^2, we see that v^i is a unit vector,

$$\begin{aligned} 1 &= \left(1 + a^2\rho^2\right)\left(\frac{d\rho}{ds}\right)^2 + \rho^2\left(\frac{d\varphi}{ds}\right)^2 \\ &= \left(1 + a^2\rho^2\right)\left(v^1\right)^2 + \rho^2\left(v^2\right)^2 \end{aligned}$$

Therefore, if we can get one component, we may easily find the other.

The second differential equation is not difficult. Rearranging and setting $v^2 = \frac{d\varphi}{ds}$, it becomes

$$\frac{1}{v^2}\frac{dv^2}{ds} = -\frac{2}{\rho}\frac{d\rho}{ds}$$

Multiplying by ds, we may integrate:

$$\begin{aligned} \frac{1}{v^2}dv^2 &= -\frac{2}{\rho}d\rho \\ \int_{v_0^2}^{v^2}\frac{1}{v^2}dv^2 &= -\int_{\rho_0}^{\rho}\frac{2}{\rho}d\rho \\ \ln\frac{v^2}{v_0^2} &= -2\ln\frac{\rho}{\rho_0} \end{aligned}$$

Then writing $-2\ln\frac{\rho}{\rho_0} = \ln\left(\frac{\rho_0}{\rho}\right)^2$ and exponentiating,

$$\begin{aligned} \frac{v^2}{v_0^2} &= \left(\frac{\rho_0}{\rho}\right)^2 \\ v^2 &= v_0^2\left(\frac{\rho_0}{\rho}\right)^2 \end{aligned}$$

The first component is therefore given by

$$\begin{aligned} 1 &= \left(1 + a^2\rho^2\right)\left(v^1\right)^2 + \rho^2\left(v^2\right)^2 \\ &= \left(1 + a^2\rho^2\right)\left(v^1\right)^2 + \rho^2\left(v_0^2\left(\frac{\rho_0}{\rho}\right)^2\right)^2 \\ &= \left(1 + a^2\rho^2\right)\left(v^1\right)^2 + \left(\frac{\rho_0^2 v_0^2}{\rho}\right)^2 \\ \left(1 + a^2\rho^2\right)\left(v^1\right)^2 &= 1 - \frac{\left(\rho_0^2 v_0^2\right)^2}{\rho^2} \\ \left(v^1\right)^2 &= \frac{1 - \frac{\left(\rho_0^2 v_0^2\right)^2}{\rho^2}}{1 + a^2\rho^2} \end{aligned}$$

so that

$$v^1 = \sqrt{\frac{1 - \frac{\left(\rho_0^2 v_0^2\right)^2}{\rho^2}}{1 + a^2\rho^2}}$$

At $\rho = \rho_0$,

$$v_0^1 = \sqrt{\frac{1 - \rho_0^2 \left(v_0^2\right)^2}{1 + a^2 \rho_0^2}}$$

5 Orbital equation

Since

$$\begin{aligned} v^1 &= \frac{d\rho}{ds} \\ v^2 &= \frac{d\varphi}{ds} \end{aligned}$$

we may derive an orbit equation, $\rho(\varphi)$ by taking the ratio,

$$\frac{v^1}{v^2} = \frac{d\rho}{d\varphi}$$

Then, substituting,

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \frac{1}{v_0^2 \left(\frac{\rho_0}{\rho}\right)^2} \sqrt{\frac{1 - \frac{\left(\rho_0^2 v_0^2\right)^2}{\rho^2}}{1 + a^2 \rho^2}} \\ &= \frac{\rho^2}{v_0^2 \left(\rho_0\right)^2} \sqrt{\frac{1 - \frac{\left(\rho_0^2 v_0^2\right)^2}{\rho^2}}{1 + a^2 \rho^2}} \\ &= \frac{\rho}{v_0^2 \left(\rho_0\right)^2} \sqrt{\frac{\rho^2 - \left(\rho_0^2 v_0^2\right)^2}{1 + a^2 \rho^2}} \end{aligned}$$

Dividing by this and multiplying by $d\varphi$, we may integrate,

$$\sqrt{\frac{1 + a^2 \rho^2}{\rho^2 - \left(\rho_0^2 v_0^2\right)^2}} \frac{d\rho}{\rho} = \frac{d\varphi}{v_0^2 \left(\rho_0\right)^2}$$

and therefore, taking $\varphi_0 = 0$,

$$\varphi = \frac{1}{v_0^2 \left(\rho_0\right)^2} \int_{\rho_0}^{\rho} \sqrt{\frac{1 + a^2 \rho^2}{\rho^2 - \left(\rho_0^2 v_0^2\right)^2}} \frac{d\rho}{\rho}$$

This is integrated quickly using Wolfram integrator to give a logarithmic dependence.