Geodesics as gravity

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It is not obvious that curvature can account for gravity. The orbiting path of a planet, for example, does not immediately seem to be the shortest path between points. Even more immediate is the way the motion of projectiles differs depending on initial conditions. Consider the tightly parabolic path followed by a stone thrown high into the air, contrasted with the nearly straight-line path of an arrow. How can these two paths, which may pass through nearly identical regions of space, be regarded as tracing geodesics in the same curved geometry? The answer lies in the importance of time in giving a 4-dimensional picture.

1 The curvature of a curve

Suppose we have a curve lying in the xy-plane, given by $y = y(x)$. At any point $P = (x_0, y(x_0))$ of the curve we place a circle. The *best fit* circle is the one which is tangent at P, and matches the curve as nearly as possible. To find this circle, choose coordinates with origin at P and the x axis tangent at P. Then $y(x_0) = y(0) = 0$, and the slope of the curve vanishes, $\left.\frac{dy}{dx}\right|_{x_0=0} = 0$. Now expand $y(x)$near P. By our choice of coordinates, the first two terms in the series vanish,

$$
\begin{aligned}
y_{curve}(x) &= y(x_0) + \left.\frac{dy}{dx}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0}(x - x_0)^2 + \cdots \\
&= \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0} x^2 + \cdots
\end{aligned}
$$

Suppose, for concreteness, that at P the curve has $\left.\frac{d^2y}{dx^2}\right|_{x_0} > 0$.

Now define a circle of radius R, tangent to the curve at P. The center of the circle will be a distance R up the y axis and consist of point satisfying

$$
x^2 + (y - R)^2 = R^2
$$

Solving this for $y_{circle}(x)$, we have

$$
y_{circle} = R \pm \sqrt{R^2 - x^2}
$$

To get points near the origin, we require the lower sign, and expand in a power series,

$$
\begin{aligned}
y_{circle} &= R - \sqrt{R^2 - x^2} \\
&= R - R\sqrt{1 - \frac{x^2}{R^2}} \\
&\approx R - R\left(1 - \frac{1}{2}\frac{x^2}{R^2} + \cdots\right) \\
&= \frac{x^2}{2R} + \cdots
\end{aligned}
$$

The best fit circle is the circle determined by matching these curves:

$$y_{curve}(x) = \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0} x^2 + \cdots$$
$$y_{circle} = \frac{x^2}{2R} + \cdots$$

In general, the higher derivatives will not match, but we can match the second derivative by choosing the radius of the circle:

$$\frac{x^2}{2R} = \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_0} x^2$$
$$\frac{1}{R} = \left.\frac{d^2y}{dx^2}\right|_{x_0}$$

We define the *curvature* at any point of the curve to be $\frac{1}{R}$ where R is the radius of the best fit circle.

2 The curvature of projectile motion

Consider the two motions: a stone is thrown upward at a steep angle θ, landing at a distance d, while another is thrown at high speed at a target in the same final location. Each stone follows a parabolic motion:

$$x = v_{0x}t$$
$$y = v_{0y}t - \frac{1}{2}gt^2$$

so that

$$y(x) = v_{0y}\left(\frac{x}{v_{0x}}\right) - \frac{1}{2}g\left(\frac{x}{v_{0x}}\right)^2$$

The distance traveled, $x = d$, occurs when $y = 0$:

$$0 = v_{0y}\left(\frac{d}{v_{0x}}\right) - \frac{1}{2}g\left(\frac{d}{v_{0x}}\right)^2$$
$$d = \frac{2v_{0x}v_{0y}}{g}$$

If the projectile is launched at an angle given by $\tan\theta = \frac{v_{0y}}{v_{0x}}$ then

$$v_{0y} = v_{0x}\tan\theta$$

and

$$d = \frac{2v_{0x}^2\tan\theta}{g}$$
$$v_{0x} = \sqrt{\frac{gd}{2\tan\theta}}$$
$$v_{0y} = v_{0x}\tan\theta$$
$$= \sqrt{\frac{1}{2}gd\tan\theta}$$

The maximum height is reached at when $v_y = 0$,

$$0 = v_{0y} - gt$$
$$t = \frac{v_{0y}}{g}$$

and at this time we have

$$\begin{aligned} h &= \frac{v_{0y}^2}{2g} \\ &= \frac{d}{4}\tan\theta \end{aligned}$$

We find the curvature at the top of the trajectory, $P = (x_{top}, h) = \left(\frac{d}{2}, \frac{d}{4}\tan\theta\right)$. At any point, the curve may be described by

$$\begin{aligned} x &= x_0 + v_{0x}t \\ y &= y_0 + v_{oy}t - \frac{1}{2}gt^2 \end{aligned}$$

At the top, $(x_0, y_0) = \left(\frac{d}{2}, \frac{d}{4}\tan\theta\right)$ and the initial velocity is $(v_{0x}, v_{0y}) = (v_{ox}, 0)$. Therefore,

$$\begin{aligned} x &= \frac{d}{2} + v_{0x}t \\ y &= \frac{d}{4}\tan\theta - \frac{1}{2}gt^2 \end{aligned}$$

and the curve $y(x)$ is

$$\begin{aligned} t &= \frac{1}{v_{0x}}\left(x - \frac{d}{2}\right) \\ &= \sqrt{\frac{2\tan\theta}{gd}}\left(x - \frac{d}{2}\right) \\ y &= \frac{d}{4}\tan\theta - \frac{1}{2}g\frac{2\tan\theta}{gd}\left(x - \frac{d}{2}\right)^2 \\ &= \frac{d}{4}\tan\theta - \frac{\tan\theta}{d}\left(x - \frac{d}{2}\right)^2 \end{aligned}$$

The best fit circle will match the second derivative:

$$\frac{1}{R} = \frac{2\tan\theta}{d}$$

and we see that the curvature is large for steep angles and small for shallow angles.

There is a dramatic difference in the curvature for different initial conditions. Suppose the total time of flight,

$$t = \frac{2v_{0y}}{g} = \sqrt{\frac{2d}{g}\tan\theta}$$

is 1 second for one stone and 10 seconds for another over a distance $d = 25$ meters. Then for the first

$$\begin{aligned} 1 &= \sqrt{\frac{50}{10}\tan\theta} \\ \frac{1}{5} &= \tan\theta \end{aligned}$$

the curvature at the top is

$$\begin{aligned} \frac{1}{R_1} &= \frac{2\tan\theta}{d} \\ &= \frac{2}{125} \\ &= .016 \end{aligned}$$

while for the second,

$$\begin{aligned} 10 &= \sqrt{\frac{50}{10}\tan\theta} \\ \tan\theta &= 20 \end{aligned}$$

and the curvature is

$$\frac{1}{R_{10}} = \frac{40}{25} = 1.6$$

The curvatures differ by a factor of 100 for ordinary trajectories. This makes is impossible to model the motions of the stones by letting them follow geodesics in a curved 3-dimensional geometry. To build a geometric model, we need the curvature to be very nearly the same for the two stones, so they both move along comparably shaped paths.

3 The curvature in spacetime

Now place the motion in spacetime. The (non-relativistic) projectile follows a curve parameterized by time:

$$(ct, x, y) = \left(ct, v_{0x}t, \frac{1}{2}gt^2\right)$$

The motion lies in a plane rotated in the xt plane at an angle with $\tan\varphi = \frac{v_{0x}}{c} \ll 1$. Let $\tau = \sqrt{c^2t^2 - x^2}$ be proper distance (proper time!) along this direction. Then

$$\begin{aligned} c\tau &= \sqrt{c^2t^2 - v_{0x}^2t^2} \\ &= ct\sqrt{1-\tan^2\varphi} \\ \tau &\approx t \end{aligned}$$

Then we may parameterize the curve in this plane by

$$\begin{aligned} (c\tau, y) &= \left(c\tau, \frac{g\tau^2}{2}\right) \\ &= \left(\lambda, \frac{g\lambda^2}{2c^2}\right) \end{aligned}$$

where we parameterize with $\lambda = c\tau$. The best-fit circle to this curve gives

$$\begin{aligned} \frac{c^2\tau^2}{2R} &= \frac{gc^2\tau^2}{2c^2} \\ \frac{1}{R} &= \frac{g}{c^2} \end{aligned}$$

and this is independent of initial conditions.

This means that the high arc of a slow stone, and the flat trajectory of a fast stone have the *same* curvature in spacetime! Both trajectories could lie on a spherically curved surface where the sphere has a huge radius, $\frac{c^2}{g}$, so it is only gently curved. In sharp contrast to the Euclidean case, where a factor of 10 difference in time of flight requires a factor of 100 difference in radius of curvature, even a difference of a factor of 10^4 in time of flight makes completely negligible difference to the curvature required in spacetime.

Exerise: Find the difference in curvatures $\frac{1}{R}$ between initial velocities of $3\frac{m}{sec}$ and $3\times 10^4\frac{m}{sec}$

Answer: If we don't neglect terms of order $\frac{v^2}{c^2}$, then t and τ are related by

$$\begin{aligned} c\tau &= \sqrt{c^2t^2 - v_{0x}^2t^2} \\ &= ct\sqrt{1-\frac{v_{0x}^2}{c^2}} \\ \tau &= t\sqrt{1-\frac{v_{0x}^2}{c^2}} \end{aligned}$$

so

$$t = \frac{\tau}{\sqrt{1-\frac{v_{0x}^2}{c^2}}}$$

Now, when we parameterize the curve by $c\tau$, we have

$$\begin{aligned} (c\tau, y) &= \left(c\tau, \frac{gt^2}{2}\right) \\ &= \left(c\tau, \frac{g}{2c^2}\left(\frac{c\tau}{\sqrt{1-\frac{v_{0x}^2}{c^2}}}\right)^2\right) \end{aligned}$$

Therefore, the curvature is given exactly by

$$\begin{aligned} \frac{c^2\tau^2}{2R} &= \frac{g}{2c^2}\frac{c^2\tau^2}{1-\frac{v_{0x}^2}{c^2}} \\ \frac{1}{R} &= \frac{g}{c^2}\frac{1}{1-\frac{v_{0x}^2}{c^2}} \end{aligned}$$

Now consider stones thrown with velocities of $3\frac{m}{sec}$ and $3\times 10^4\frac{m}{sec}$. In the first case,

$$\begin{aligned} \frac{1}{1-\frac{v_{0x}^2}{c^2}} &= \frac{1}{1-\frac{9}{9\times 10^{16}}} \\ &= \frac{1}{1-10^{-16}} \\ &\approx 1+10^{-16} \end{aligned}$$

so the curvature is

$$\begin{aligned} \frac{1}{R} &= \frac{g}{c^2}\frac{1}{1-\frac{v_{0x}^2}{c^2}} \\ &= \frac{9.8}{9\times 10^{16}}\times\left(1+10^{-16}\right) \\ &\approx 1.1\times 10^{-16} + 1.1\times 10^{-32} \end{aligned}$$

For the second stone,

$$\begin{aligned} \frac{1}{1-\frac{v_{0x}^2}{c^2}} &= \frac{1}{1-\frac{9\times 10^{8}}{9\times 10^{16}}} \\ &= \frac{1}{1-10^{-8}} \\ &\approx 1+10^{-8} \end{aligned}$$

and the curvature is

$$
\begin{aligned}
\frac{1}{R} &= \frac{g}{c^2}\frac{1}{1-\frac{v_{0x}^2}{c^2}} \\
&= \frac{9.8}{9\times 10^{16}} \times \left(1+10^{-8}\right) \\
&\approx 1.1\times 10^{-16} + 1.1\times 10^{-24}
\end{aligned}
$$

The difference between these negligible.