The geodesic equation as Newton's Law of Universal Gravitation

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1 The geodesic equation as Newton's second law

We have seen that curves of extremal proper length or time satisfy the geodesic equation,

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\ \mu\beta} u^\mu u^\beta \tag{1}$$

where $u^\alpha = \frac{dx^\alpha}{d\tau}$ is the 4-velocity. Since the acceleration may be written using the chain rule as

$$\begin{aligned} \frac{d^2x^\alpha}{d\tau^2} &= \frac{du^\alpha}{d\tau} \\ &= \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta} \\ &= u^\beta \frac{\partial u^\alpha}{\partial x^\beta} \end{aligned}$$

we may write the geodesic equation as

$$\begin{aligned} 0 &= u^\beta \frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\ \mu\beta} u^\mu u^\beta \\ &= u^\beta \left(\frac{\partial u^\alpha}{\partial x^\beta} + u^\mu \Gamma^\alpha_{\ \mu\beta} \right) \\ &= u^\beta D_\beta u^\alpha \end{aligned}$$

This describes an *autoparallel*: the parallel transport of the vector u^α along its own direction.

In the form of eq.(1), the geodesic equation gives the acceleration of a particle in terms of geometric quantities. This suggests that it might be possible to explain some force using the connection. If we multiply the geodesic equation by the mass of a particle, the left side becomes the proper time rate of change of the momentum,

$$\frac{dp^\alpha}{d\tau} = -m\Gamma^\alpha_{\ \mu\beta} u^\mu u^\beta$$

where $p^\alpha = mu^\alpha$. The question is, can we make the connection term look like a force?

2 Nearly flat space

Suppose the metric is nearly that of a flat space, differing only in the time component,

$$g_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -c^2+\phi & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and suppose the function ϕ depends only on $r = \sqrt{x^2+y^2+z^2}$. Then the interval of proper time is

$$d\tau^2 = \left(1 - \frac{1}{c^2}\phi\right) dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right)$$

We would like to find the connection, which is built from derivatives of $g_{\mu\nu}$. The only nonvanishing derivatives of the metric are

$$g_{00,i} = \begin{pmatrix} \phi_{,i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $i = 1, 2, 3$ and

$$\begin{aligned} \phi_{,i} &= \frac{d\phi}{dr}\frac{\partial r}{\partial x^i} \\ &= \frac{d\phi}{dr}\frac{x_i}{r} \end{aligned}$$

This means the only nonvanishing connection components must have two time indices, 0, and one spatial one, i:

$$\begin{aligned} \Gamma_{00i} = \Gamma_{0i0} &= \frac{1}{2}\left(g_{00,i} + g_{0i,0} - g_{0i,0}\right) \\ &= \frac{1}{2}\frac{d\phi}{dr}\frac{x_i}{r} \\ \Gamma_{i00} &= \frac{1}{2}\left(g_{i0,0} + g_{i0,0} - g_{00,i}\right) \\ &= -\frac{1}{2}\frac{d\phi}{dr}\frac{x_i}{r} \end{aligned}$$

Raising the first index on each, we have

$$\begin{aligned} \Gamma^0_{\ 0i} = \Gamma^0_{\ i0} &= g^{0\alpha}\Gamma_{\alpha i0} \\ &= g^{00}\Gamma_{0i0} \\ &= -\frac{1}{2(1-\phi)}\frac{d\phi}{dr}\frac{x_i}{r} \end{aligned}$$

and

$$\begin{aligned} \Gamma^i_{\ 00} &= g^{i\alpha}\Gamma_{\alpha 00} \\ &= g^{ij}\Gamma_{j00} \\ &= \frac{1}{2}\frac{d\phi}{dr}\frac{x^i}{r} \end{aligned}$$

Now return to the geodesic equation. Writing the 4-velocity as $u^\alpha = \gamma\left(c, v^i\right)$ and computing each component of the equation separately, we have

$$\frac{dp^0}{d\tau} = -m\Gamma^0_{\ \mu\beta}u^\mu u^\beta$$

$$
\begin{aligned}
&= -m\Gamma^0_{\ 00}u^0u^0 - m\Gamma^0_{\ 0i}u^0u^i - m\Gamma^0_{\ i0}u^iu^0 - m\Gamma^0_{\ ij}u^iu^j \\
&= -2m\Gamma^0_{\ 0i}u^0u^i \\
&= -2m\left(-\frac{1}{2(1-\phi)}\frac{d\phi}{dr}\frac{x_i}{r}\right)\gamma^2 cv^i \\
&= \frac{mc}{r(1-\phi)}\gamma^2\frac{d\phi}{dr}v^ix_i
\end{aligned}
$$

for the time component and

$$
\begin{aligned}
\frac{dp^i}{d\tau} &= -m\Gamma^i_{\ \mu\beta}u^\mu u^\beta \\
&= -m\Gamma^i_{\ 00}u^0u^0 - m\Gamma^i_{\ 0j}u^0u^j - m\Gamma^i_{\ j0}u^ju^0 - m\Gamma^i_{\ jk}u^ju^k \\
&= -m\Gamma^i_{\ 00}u^0u^0 \\
&= -\frac{1}{2}m\gamma^2c^2\frac{d\phi}{dr}\frac{x^i}{r}
\end{aligned}
$$

We consider the case of non-relativistic velocities, so that we may set $\gamma \approx 1$. Then, noting that $\frac{x^2}{r}$ is just a unit vector in the radial direction, we can choose ϕ so that this last expression looks like the Newton's gravitational force,

$$
\begin{aligned}
-\frac{1}{2}mc^2\frac{d\phi}{dr}\hat{\mathbf{r}} &= -\frac{GMm}{r^2}\hat{\mathbf{r}} \\
\frac{d\phi}{dr} &= \frac{2GM}{r^2c^2}
\end{aligned}
$$

Integrating,

$$
\phi = -\frac{2GM}{rc^2}
$$

Notice that $\frac{2GM}{r}$ is just the square of the classical escape velocity,

$$
\begin{aligned}
\frac{1}{2}mv^2 &= \frac{GMm}{r} \\
v^2 &= \frac{2GM}{r}
\end{aligned}
$$

so the magnitude of ϕ is very small, $\frac{v^2_{escape}}{c^2} \ll 1$ for planets and ordinary stars. With this choice for ϕ, the metric differs from flat space by only a term of order $\frac{v^2_{escape}}{c^2}$, and we reproduce Newton's law of gravity,

$$
\frac{d\mathbf{p}}{dt} = -\frac{GMm}{r^2}\hat{\mathbf{r}}
$$

at normal velocities $v \ll c$.

Now consider the time component of the geodesic equation. Substituting this result for ϕ into the 0 component of the geodesic equation,

$$
\frac{dp^0}{d\tau} = \frac{1}{1+\frac{2GM}{rc^2}}\frac{2GMm}{r^2c}\mathbf{v}\cdot\hat{\mathbf{r}}
$$

Since $p^0 = \frac{E}{c}$ and $\tau \approx t$,

$$
\frac{dE}{dt} = \frac{1}{1+\frac{2GM}{rc^2}}\frac{2GMm}{r^2}\mathbf{v}\cdot\hat{\mathbf{r}}
$$

$$
\begin{aligned}
&= \frac{1}{1+\frac{v_{escape}^2}{c^2}} \frac{2GMm}{r^2} \mathbf{v}\cdot\hat{\mathbf{r}} \\
&\approx \frac{2GMm}{r^2}\hat{\mathbf{r}}\cdot\mathbf{v} \\
&= 2\mathbf{F}_{Newton}\cdot\mathbf{v}
\end{aligned}
$$

This differs from the actual rate of change of energy of a falling particle by a factor of 2. This factor is eliminated when we use a more realistic metric.

Exercise: Repeat these arguments using the metric

$$
g_{\mu\nu} = \begin{pmatrix} -c^2+\phi & 0 & 0 & 0 \\ 0 & 1+\lambda\phi & 0 & 0 \\ 0 & 0 & 1+\lambda\phi & 0 \\ 0 & 0 & 0 & 1+\lambda\phi \end{pmatrix}
$$

where λ is constant. Can you choose λ to get the correct expression for the change in energy?