Parallel transport and geodesics

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1 Parallel transport

Before defining a general notion of curvature for an arbitrary space, we need to know how to compare vectors at different positions on a manifold. Parallel transport provides a way to compare a vector in one tangent plane to a vector in another, by moving the vector along a curve without changing it. Suppose we have a curve with unit tangent $\mathbf{t}$ in flat space and Cartesian coordinates. Then to move a vector $\mathbf{v}$ along this curve without changing it simply means holding the components constant. At any point λ along the curve, we may find the transported components by solving

$$(\mathbf{t} \cdot \boldsymbol{\nabla}) \mathbf{v} = 0$$

or, in components,

$$t^i \partial_i v^j = 0$$

Since the connection vanishes in Cartesian coordinates, this is the same as writing

$$t^i D_i v^j = 0$$

but this expression now holds in any coordinates. The same argument holds in a curved space because close enough to any point we may find Cartesian coordinates, transport infinitesimally, then change coordinates to Cartesian again. At each point, the Cartesian expression may be written covariantly, but the covariant expression is the same at every point of the curve regardless of coordinates. We therefore define parallel transport of a vector v^α along a curve with tangent $u^\alpha(\lambda) = \frac{dx^\alpha}{d\lambda}$ to be the solution $v^\alpha(\lambda)$ to the equation

$$u^\alpha D_\alpha v^\beta = 0$$

A curve is called *autoparallel* if it is transported along its own direction,

$$v^\alpha D_\alpha v^\beta = 0$$

Notice that parallel transport preserves the length of the vector because

$$\begin{aligned}
u^\alpha D_\alpha \left(|\mathbf{v}|^2 \right) &= u^\alpha D_\alpha \left(g_{\mu\nu} v^\mu v^\nu \right) \\
&= u^\alpha \left(D_\alpha g_{\mu\nu} v^\mu v^\nu + g_{\mu\nu} D_\alpha v^\mu v^\nu + g_{\mu\nu} v^\mu D_\alpha v^\nu \right) \\
&= g_{\mu\nu} \left(u^\alpha D_\alpha v^\mu \right) v^\nu + g_{\mu\nu} v^\mu \left(u^\alpha D_\alpha v^\nu \right) \\
&= 0
\end{aligned}$$

2 Example: parallel transport on the 2-sphere

Consider the parallel transport of a vector around a $\theta = \theta_0$ curve on the 2-sphere. The curve itself may parameterized using φ as $x^i = (\theta_0, \varphi)$, with tangent, $t^i = (0, 1)$. The length of this tangent vector is given by

$$\begin{aligned} l^2 &= g_{ij}t^i t^j \\ &= R^2 \sin^2\theta_0 \end{aligned}$$

so the unit tangent is

$$u^i = \frac{1}{R\sin\theta_0}(0,1)$$

At $\varphi = 0$, let $v^i = \left(v_0^\theta, v_0^\varphi\right)$, and solve the parallel transport equation for $v^i(\varphi)$,

$$\begin{aligned} 0 &= u^i D_i v^j \\ &= \frac{1}{R\sin\theta_0} D_\varphi v^j \\ 0 &= \partial_\varphi v^j + v^k \Gamma^j_{\ k\varphi} \end{aligned}$$

2.1 The metric

Since the metric is given by the line element, $ds^2 = R^2 d\theta^2 + \sin^2\theta d\varphi^2$, we have, in matrix components,

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

with inverse

$$g^{ij} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}$$

There are intrinsic ways to get this metric. One approach is to specify the symmetries we require – three independent rotations. There are techniques for finding the most general metric with given symmetry, so we can derive this form directly. Alternatively, we could ask for 2-dim spaces of constant curvature. Computing the metric for a general 2-geometry, then imposing constant curvature gives a set of differential equations that will lead to this form.

2.2 The connection

Since the only non-constant component of the metric tensor is $g_{\varphi\varphi}$, there are only three nonvanishing connection components,

$$\begin{aligned} \Gamma^\theta_{\ \varphi\varphi} &= \frac{1}{2}g^{\theta\theta}\left(g_{\theta\varphi,\varphi} + g_{\theta\varphi,\varphi} - g_{\varphi\varphi,\theta}\right) \\ &= \frac{1}{2}\frac{1}{R^2}\left(-\left(R^2\sin^2\theta\right)_{,\theta}\right) \\ &= -\sin\theta\cos\theta \\ \Gamma^\varphi_{\ \theta\varphi} = \Gamma^\varphi_{\ \varphi\theta} &= \frac{1}{2}g^{\varphi\varphi}\left(g_{\varphi\varphi,\theta} + g_{\varphi\theta,\varphi} - g_{\theta\varphi,\varphi}\right) \\ &= \frac{1}{2}\frac{1}{R^2\sin^2\theta}\left(R^2\sin^2\theta\right)_{,\theta} \\ &= \frac{\cos\theta}{\sin\theta} \end{aligned}$$

2.3 Parallel transport

The parallel transport equation becomes

$$\begin{aligned} 0 &= \partial_\varphi v^j + v^k\Gamma^j_{\ k\varphi} \\ &= \partial_\varphi v^j + v^\theta\Gamma^j_{\ \theta\varphi} + v^\varphi\Gamma^j_{\ \varphi\varphi} \end{aligned}$$

There are two components to check. For $j = \theta$ we have

$$\begin{aligned} 0 &= \partial_\varphi v^\theta + v^\varphi \Gamma^\theta_{\varphi\varphi} \\ &= \frac{\partial v^\theta}{\partial \varphi} - v^\varphi \sin\theta_0 \cos\theta_0 \end{aligned}$$

For $j = \varphi$,

$$\begin{aligned} 0 &= \partial_\varphi v^\varphi + v^\theta \Gamma^\varphi_{\theta\varphi} \\ &= \frac{\partial v^\varphi}{\partial \varphi} + v^\theta \frac{\cos\theta_0}{\sin\theta_0} \end{aligned}$$

Therefore, we need to solve the coupled equations,

$$\begin{aligned} 0 &= \frac{\partial v^\theta}{\partial \varphi} - v^\varphi \sin\theta_0 \cos\theta_0 \\ 0 &= \frac{\partial v^\varphi}{\partial \varphi} + v^\theta \frac{\cos\theta_0}{\sin\theta_0} \end{aligned}$$

Taking a second derivative of the first equation and substituting the second,

$$\begin{aligned} 0 &= \frac{\partial^2 v^\theta}{\partial \varphi^2} - \frac{\partial v^\varphi}{\partial \varphi} \sin\theta_0 \cos\theta_0 \\ &= \frac{\partial^2 v^\theta}{\partial \varphi^2} + v^\theta \frac{\cos\theta_0}{\sin\theta_0} \sin\theta_0 \cos\theta_0 \\ &= \frac{\partial^2 v^\theta}{\partial \varphi^2} + v^\theta \cos^2\theta_0 \end{aligned}$$

Similarly, differentiating the second equation and substituting the first we have

$$\begin{aligned} 0 &= \frac{\partial^2 v^\varphi}{\partial \varphi^2} + \frac{\partial v^\theta}{\partial \varphi} \frac{\cos\theta_0}{\sin\theta_0} \\ &= \frac{\partial^2 v^\varphi}{\partial \varphi^2} + v^\varphi \sin\theta_0 \cos\theta_0 \frac{\cos\theta_0}{\sin\theta_0} \\ &= \frac{\partial^2 v^\varphi}{\partial \varphi^2} + v^\varphi \cos^2\theta_0 \end{aligned}$$

Each of these is just the equation for sinusoidal oscillation, so we may immediately write the solution,

$$\begin{aligned} v^\theta(\varphi) &= A\cos\alpha\varphi + B\sin\alpha\varphi \\ v^\varphi(\varphi) &= C\cos\alpha\varphi + D\sin\alpha\varphi \end{aligned}$$

with the frequency α given by

$$\alpha = \cos\theta_0$$

Starting the curve at $\varphi = 0$, it will close at $\varphi = 2\pi$. Then for v^α we have the initial condition $v^\alpha(0) = \left(v_0^\theta, v_0^\varphi\right)$, and from the original differential equations we must have

$$\begin{aligned} \left.\frac{\partial v^\theta}{\partial \varphi}\right|_{\varphi=0} &= v_0^\varphi \sin\theta_0 \cos\theta_0 \\ \left.\frac{\partial v^\varphi}{\partial \varphi}\right|_{\varphi=0} &= -v_0^\theta \frac{\cos\theta_0}{\sin\theta_0} \end{aligned}$$

These conditions determine the constants A, B, C, D to be

$$\begin{aligned}
v^{\theta}(\varphi) &= v_0^{\theta} \cos\alpha\varphi + \frac{v_0^{\varphi} \sin\theta_0 \cos\theta_0}{\alpha} \sin\alpha\varphi \\
&= v_0^{\theta} \cos\alpha\varphi + v_0^{\varphi} \sin\theta_0 \sin\alpha\varphi \\
v^{\varphi}(\varphi) &= v_0^{\varphi} \cos\alpha\varphi - \frac{v_0^{\theta}}{\sin\theta_0} \sin\alpha\varphi
\end{aligned}$$

This gives the form of the transported vector at any point around the circle,

$$\begin{aligned}
v^{\theta}(\varphi) &= v_0^{\theta} \cos(\varphi\cos\theta_0) + v_0^{\varphi} \sin\theta_0 \sin(\varphi\cos\theta_0) \\
v^{\varphi}(\varphi) &= v_0^{\varphi} \cos(\varphi\cos\theta_0) - \frac{v_0^{\theta}}{\sin\theta_0} \sin(\varphi\cos\theta_0)
\end{aligned}$$

Look at the inner product of $\mathbf{v}$ with the tangent vector at the same point, $\mathbf{u} = \frac{1}{R\sin\theta_0}(0, 1)$,

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= g_{\varphi\varphi} t^{\varphi} \left(v_0^{\varphi} \cos(\varphi\cos\theta_0) - \frac{v_0^{\theta}}{\sin\theta_0} \sin(\varphi\cos\theta_0) \right) \\
&= R\sin\theta_0 \left(v_0^{\varphi} \cos(\varphi\cos\theta_0) - \frac{v_0^{\theta}}{\sin\theta_0} \sin(\varphi\cos\theta_0) \right) \\
&= v_0^{\varphi} R\sin\theta_0 \cos(\varphi\cos\theta_0) - v_0^{\theta} R \sin(\varphi\cos\theta_0)
\end{aligned}$$

If the circle is at the equator, $\theta_0 = \frac{\pi}{2}$, then

$$\mathbf{u}\cdot\mathbf{v} = v_0^{\varphi} R$$

is constant. On the other hand, near the pole, $\theta_0 \ll 1$,

$$\mathbf{u}\cdot\mathbf{v} = v_0^{\varphi} R\theta_0 \cos\varphi - v_0^{\theta} R \sin\varphi$$

and the transported vector rotates almost completely around the tangent.

3 Geodesics

Consider a curve, $x^{\alpha}(\lambda)$ in an arbitrary (possibly curved) spacetime, with the proper interval given by

$$d\tau^2 = -g_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

Then the 4-velocity along the curve is given by

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

and in an arbitrary parameterization, the tangent is $t^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$. Then proper time (or length) along the curve is given by integrating

$$\begin{aligned}
\tau &= \int_0^{\tau} \sqrt{-g_{\alpha\beta} dx^{\alpha} dx^{\beta}} \\
&= \int_0^{\tau} \sqrt{-g_{\alpha\beta} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda}} d\lambda
\end{aligned}$$

A curve of extremal proper length is called a geodesic. We may find an equation for geodesics by finding the equation for the extrema of τ,

$$\begin{aligned}
0 &= \delta\tau \\
&= \delta\int_0^\tau \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}d\lambda \\
&= -\int_0^\tau \frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\delta\left(g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)d\lambda \\
&= -\int_0^\tau \frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\left(\delta g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}+g_{\alpha\beta}\frac{d\delta x^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}+g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{d\delta x^\beta}{d\lambda}\right)d\lambda \\
&= -\int_0^\tau \frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}\left(g_{\alpha\beta,\mu}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right)\delta x^\mu d\lambda \\
&\quad +\int_0^\tau \frac{d}{d\lambda}\left(\frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}g_{\alpha\beta}\frac{dx^\beta}{d\lambda}\right)\delta x^\alpha d\lambda \\
&\quad +\int_0^\tau \frac{d}{d\lambda}\left(\frac{1}{2\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}}g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\right)\delta x^\beta d\lambda
\end{aligned}$$

Now choose the parameter λ to be proper time (length) so that

$$\begin{aligned}
g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} &= g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\
&= -c^2 \\
&= -1
\end{aligned}$$

Then we have

$$\begin{aligned}
0 &= \frac{1}{2}\int_0^\tau \left(\left(-g_{\alpha\beta,\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)\delta x^\mu+\frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\beta}{d\tau}\right)\delta x^\alpha+\frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\right)\delta x^\beta\right)d\lambda \\
&= \frac{1}{2}\int_0^\tau \left(\left(-g_{\alpha\beta,\mu}u^\alpha u^\beta\right)\delta x^\mu+\frac{d}{d\tau}\left(g_{\alpha\beta}u^\beta\right)\delta x^\alpha+\frac{d}{d\tau}\left(g_{\alpha\beta}u^\alpha\right)\delta x^\beta\right)d\lambda \\
&= \frac{1}{2}\int_0^\tau \left(\left(-g_{\alpha\beta,\mu}u^\alpha u^\beta\right)\delta x^\mu+\left(g_{\alpha\beta,\nu}\frac{dx^\nu}{d\tau}u^\beta+g_{\alpha\beta}\frac{du^\beta}{d\tau}\right)\delta x^\alpha+\left(g_{\alpha\beta,\nu}\frac{dx^\nu}{d\tau}u^\alpha+g_{\alpha\beta}\frac{du^\alpha}{d\tau}\right)\delta x^\beta\right)d\lambda \\
&= \frac{1}{2}\int_0^\tau \left(\left(-g_{\alpha\beta,\mu}u^\alpha u^\beta\right)+\left(g_{\mu\beta,\nu}u^\nu u^\beta+g_{\mu\beta}\frac{du^\beta}{d\tau}\right)+g_{\alpha\mu,\nu}u^\nu u^\alpha+g_{\alpha\mu}\frac{du^\alpha}{d\tau}\right)\delta x^\mu d\lambda \\
&= \frac{1}{2}\int_0^\tau \left(g_{\mu\beta,\nu}u^\nu u^\beta+g_{\alpha\mu,\nu}u^\nu u^\alpha-g_{\alpha\beta,\mu}u^\alpha u^\beta+g_{\mu\beta}\frac{du^\beta}{d\tau}+g_{\alpha\mu}\frac{du^\alpha}{d\tau}\right)\delta x^\mu d\lambda
\end{aligned}$$

The equation for the geodesic is therefore

$$0 = \frac{1}{2}\left(g_{\mu\beta,\nu}u^\nu u^\beta+g_{\alpha\mu,\nu}u^\nu u^\alpha-g_{\alpha\beta,\mu}u^\alpha u^\beta+g_{\mu\beta}\frac{du^\beta}{d\tau}+g_{\alpha\mu}\frac{du^\alpha}{d\tau}\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(g_{\mu\beta,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right) u^\alpha u^\beta + g_{\mu\beta}\frac{du^\beta}{d\tau} \\
0 &= g^{\mu\nu} g_{\mu\beta}\frac{du^\beta}{d\tau} + \frac{1}{2} g^{\mu\nu}\left(g_{\mu\beta,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right) u^\alpha u^\beta \\
0 &= \frac{du^\nu}{d\tau} + \Gamma^\nu{}_{\alpha\beta} u^\alpha u^\beta
\end{aligned}
$$

But this is just

$$
\begin{aligned}
\frac{du^\nu}{d\tau} + \Gamma^\nu{}_{\alpha\beta} u^\alpha u^\beta &= u^\mu \left(\frac{du^\nu}{dx^\mu} + u^\alpha \Gamma^\nu{}_{\alpha\mu}\right) \\
&= u^\mu D_\mu u^\nu
\end{aligned}
$$

and we have the equation for an autoparallel,

$$
u^\mu D_\mu u^\nu = \frac{du^\nu}{d\tau} + \Gamma^\nu{}_{\alpha\beta} u^\alpha u^\beta = 0
$$

4 Example: Geodesics on the 2-sphere

Once again consider the 2-sphere, but not look for autoparallels. We have the connection components,

$$
\begin{aligned}
\Gamma^\theta{}_{\varphi\varphi} &= -\sin\theta\cos\theta \\
\Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} &= \frac{\cos\theta}{\sin\theta}
\end{aligned}
$$

and the equations to solve are now,

$$
\begin{aligned}
\frac{du^\theta}{d\tau} + \Gamma^\theta{}_{\alpha\beta} u^\alpha u^\beta &= 0 \\
\frac{du^\varphi}{d\tau} + \Gamma^\varphi{}_{\alpha\beta} u^\alpha u^\beta &= 0
\end{aligned}
$$

Expanding the first, there is only one nonvanishing connection term,

$$
\begin{aligned}
\frac{du^\theta}{d\tau} + \Gamma^\theta{}_{\varphi\varphi} u^\varphi u^\varphi &= 0 \\
\frac{du^\theta}{d\tau} - \left(u^\varphi\right)^2 \sin\theta\cos\theta &= 0
\end{aligned}
$$

For the second,

$$
\begin{aligned}
\frac{du^\varphi}{d\tau} + \Gamma^\varphi{}_{\alpha\beta} u^\alpha u^\beta &= 0 \\
\frac{du^\varphi}{d\tau} + \Gamma^\varphi{}_{\varphi\theta} u^\varphi u^\theta + \Gamma^\varphi{}_{\theta\varphi} u^\theta u^\varphi &= 0 \\
\frac{du^\varphi}{d\tau} + 2u^\varphi u^\theta \frac{\cos\theta}{\sin\theta} &= 0
\end{aligned}
$$

Let the initial conditions be

$$
\begin{aligned}
\theta_0 &= \frac{\pi}{2} \\
\varphi_0 &= 0 \\
u_0^\varphi &= 1 \\
u_0^\theta &= 0
\end{aligned}
$$

Since every point and direction on the sphere are equivalent, there is no loss of generality in this choice. Then we initially have

$$\begin{aligned}\left(\frac{du^\theta}{d\tau}\right)_0 &= \left(u_0^\varphi\right)^2 \sin\theta_0 \cos\theta_0 \\ &= 0\end{aligned}$$

and u^θ does not change. For the φ equation, it follows that

$$\begin{aligned}\frac{du^\varphi}{d\tau} &= -2u^\varphi u^\theta \frac{\cos\theta}{\sin\theta} \\ &= 0\end{aligned}$$

so u^φ is also constant and the velocity vector is

$$u^i = (0, 1)$$

Integrating to find the curve,

$$\begin{aligned}\frac{d\theta}{d\tau} &= 0 \\ \frac{d\varphi}{d\tau} &= 1\end{aligned}$$

so $\theta = \theta_0 = \frac{\pi}{2}$ and $\varphi = \varphi_0 + \tau = \tau$. The curve is therefore the equator,

$$(\theta, \varphi) = (0, \tau)$$

We may characterize the equator as the intersection of the unique plane normal to the surface, containing the initial velocity vector. Such a plane always passes through the center of the sphere, so all geodesics are given by great circles.