Gaussian curvature

February 22, 2015

Curvature may be defined as the rate at which the amount a vector rotates when transported around an infinitesmal closed loop with respect to the area enclosed by the loop, evaluated at a point. This agrees with Gauss's definition for two surfaces, given in terms of the curvature of curves.

1 Gaussian curvature

Gauss developed a way to characterize cuvature for 2-dimensional spaces embeded in R^3, and showed that it can be computed without reference to the embedding.

1.1 The curvature of a curve

Previously, we defined the curvature of a planar curve as the inverse radius of the best fit circle. Here we generalize this definition to curves in three dimensions. To begin, let the curve be given by $\mathbf{x}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$ in Euclidean 3-space where the parameter λ is arclength. The tangent vector to the curve is given by $\mathbf{n} = \frac{d\mathbf{x}}{d\lambda}$. This will always be a unit vector since we choose arclength as the parameter. We define a unique direction orthogonal to $\mathbf{n}$ by taking another derivative,

$$\mathbf{m} \equiv \frac{1}{\kappa}\frac{d\mathbf{n}}{d\lambda}$$

where we choose κ to make $\mathbf{m}$ a unit vector. Notice that since $\mathbf{n}\cdot\mathbf{n} = 1$, differentiating shows that $\mathbf{m}$ is orthogonal to $\mathbf{n}$,

$$\mathbf{n}\cdot\frac{d\mathbf{n}}{d\lambda} = 0$$

The cross product, $\mathbf{k} = \mathbf{n}\times\mathbf{m}$ gives a third unit vector orthogonal to $\mathbf{n}$ and $\mathbf{m}$. We take the best fit circle to lie in the $\mathbf{n},\mathbf{m}$ plane with its center on the line determined by $\mathbf{m}$. The circle will also have unit tangent vector $\mathbf{n}$.

To describe the best fit circle, we expand both the circle and the curve near a point P of the curve. Let $\mathbf{x}(0) = P$. Then to second order near P the curve is

$$\begin{aligned}\mathbf{x}(\lambda) &= \mathbf{x}(0) + \frac{d\mathbf{x}(0)}{d\lambda}\lambda + \frac{1}{2!}\frac{d^2\mathbf{x}(0)}{d\lambda^2}\lambda^2 \\ &= \mathbf{x}(0) + \mathbf{n}(0)\lambda + \frac{1}{2!}\kappa\mathbf{m}\lambda^2\end{aligned}$$

The circle has its center at $\mathbf{x}(0) + R\mathbf{m}$ and lies in the $\mathbf{n},\mathbf{m}$ plane. Therefore, all points on the circle are described by points

$$\mathbf{x}_{circle}(x, y) = (\mathbf{x}(0) + R\mathbf{m}) + x\mathbf{n} + y\mathbf{m}$$

where $x^2 + y^2 = R^2$. We may parameterize the circle by setting

$$\begin{aligned}x &= R\sin\frac{\xi}{R} \\ y &= -R\cos\frac{\xi}{R}\end{aligned}$$

so that

$$\mathbf{x}_{circle}\left(\xi\right) = \left(\mathbf{x}\left(0\right) + R\mathbf{m}\right) + \mathbf{n}R\sin\frac{\xi}{R} - \mathbf{m}R\cos\frac{\xi}{R}$$

These are chosen so that ξ measures arc length on the circle (note that $\frac{d\mathbf{x}_{circle}(0)}{d\xi} = \mathbf{n}$ so the tangents coincide) and $\mathbf{x}_{circle}\left(0\right) = \mathbf{x}\left(0\right) = P$.

To second order, the circle and the curve coincide, and both ξ and λ measure arc length, so to second order, $\xi = \lambda$. Therefore, expanding the sine and cosine in $\mathbf{x}_{circle}\left(\lambda\right)$ to second order,

$$\begin{aligned}\mathbf{x}_{circle}\left(\lambda\right) &= \left(\mathbf{x}\left(0\right) + R\mathbf{m}\right) + \mathbf{n}\lambda - \mathbf{m}R\left(1 - \frac{1}{2!}\left(\frac{\lambda}{R}\right)^2\right) \\ &= \mathbf{x}\left(0\right) + \mathbf{n}\lambda + \frac{1}{2!}\frac{1}{R}\lambda^2\mathbf{m}\end{aligned}$$

and equating the curves

$$\begin{aligned}\mathbf{x}\left(\lambda\right) &= \mathbf{x}_{circle}\left(\lambda\right) \\ \mathbf{x}\left(0\right) + \mathbf{n}\left(0\right)\lambda + \frac{1}{2!}\kappa\mathbf{m}\lambda^2 &= \mathbf{x}\left(0\right) + \mathbf{n}\lambda + \frac{1}{2!}\frac{1}{R}\lambda^2\mathbf{m}\end{aligned}$$

we see that κ is the curvature of the best fit circle.

The curvature of the curve is therefore defined as the magnitude of the rate of change of this unit normal,

$$\kappa = \left|\frac{d\mathbf{n}\left(\lambda\right)}{d\lambda}\right|$$

Exercise: Find the curvature of the spiral, $\mathbf{x}\left(\lambda\right) = \frac{1}{\sqrt{R^2+1}}\left(R\cos\lambda, R\sin\lambda, \lambda\right)$ at every point.

1.2 Gaussian curvature of a 2-surface

Now consider a 2-dimensional surface embedded in Euclidean 3-space. We define the Gaussian curvature of a 2-surface as follows.

At any point, $\mathcal{P}$, of a surface, S, consider the normal, $\mathbf{n}$, to the surface. Choose any plane P containing this vector. Any two such planes are related by the angle, φ, between them, while they intersect in the line containing $\mathbf{n}$. We may therefore label all planes containing $\mathbf{n}$ by φ, giving $P\left(\varphi\right)$. The intersection of the surface S with any one of these planes will be a curve, $C\left(\varphi\right)$, lying in $P\left(\varphi\right)$. There is a unique circle in the plane $P\left(\varphi\right)$ which (a) passes through $\mathcal{P}$, (b) is tangent to $C\left(\varphi\right)$, and (c) has curvature $\kappa\left(\varphi\right)$ matching $C\left(\varphi\right)$. This is called the osculating (i.e., kissing), or best fit, circle.

The curvatures of the full set of osculating circles give a bounded function, $\kappa\left(\varphi\right)$ on a bounded interval, $[0, 2\pi]$. The function therefore has a maximum and a minimum, κ_1 and κ_2 respectively called the *principal curvatures*. The intrinsic curvature of the surface is defined as the product of the principal curvatures,

$$\mathcal{R} \equiv \kappa_1\kappa_2$$

There is another quantity, the *extrinsic curvature* defined as the sum, $\kappa_1 + \kappa_2$. The intrinsic curvature may be calculated knowing only the metric of the 2-surface, but the extrinsic curvature depends on how the shape is embedded in 3-dimensions. For example, a plane and a cylinder both have zero intrinsic curvature (cutting the cylinder, it flattens into a plane) but while the plane has $\kappa_1 = \kappa_2 = 0$, the cylinder has $\kappa_1 = \frac{1}{R}, \kappa_2 = 0$, hence nonzero extrinsic curvature.

We compute the Gaussian curvature of the 2-sphere. It is easy to see that for the 2-sphere, the principal curvatures are equal to one another, since the intersection of a plane normal to the sphere always gives a great circle. Now consider a great circle through the north pole – any curve of constant φ will do. In the embedding 3-space, taking $\varphi = 0$, the curve is $x^i = (R\sin\theta, 0, R\cos\theta) = \left(R\sin\frac{s}{R}, 0, R\cos\frac{s}{R}\right)$, where s is the arclength. The unit tangent is therefore

$$\begin{aligned}\mathbf{n} &= \frac{dx^i}{ds} \\ &= \left(\cos\frac{s}{R}, 0, -\sin\frac{s}{R}\right)\end{aligned}$$

At the north pole, $s = 0$, and the tangent points in the x-direction as expected. The principle curvature, $\kappa_1 = \kappa_2$, is given by the magnitude of

$$\begin{aligned}\frac{d\mathbf{n}}{ds} &= \frac{1}{R}\left(-\sin\frac{s}{R}, 0, -\cos\frac{s}{R}\right) \\ &= \frac{1}{R}\left(-\hat{\mathbf{k}}\right)\end{aligned}$$

so we have $\kappa_1 = \kappa_2 = \frac{1}{R}$ and the Gaussian curvature is

$$\mathcal{R} = \frac{1}{R^2}$$

which agrees with our previous result.

1.3 Gaussian curvature in the language of manifolds

The surface is a manifold, and may be described parametrically using two parameters,

$$\mathbf{x}(\lambda, \xi) = (x(\lambda, \xi), y(\lambda, \xi), z(\lambda, \xi))$$

and this provides a chart at each point of the manifold. The metric of the surface is

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x}$$

where

$$d\mathbf{x} = \left(\frac{\partial x}{\partial\lambda}d\lambda + \frac{\partial x}{\partial\xi}d\xi, \frac{\partial y}{\partial\lambda}d\lambda + \frac{\partial y}{\partial\xi}d\xi, \frac{\partial z}{\partial\lambda}d\lambda + \frac{\partial z}{\partial\xi}d\xi\right)$$

Then

$$\begin{aligned}ds^2 &= \left(\frac{\partial x}{\partial\lambda}d\lambda + \frac{\partial x}{\partial\xi}d\xi\right)^2 + \left(\frac{\partial y}{\partial\lambda}d\lambda + \frac{\partial y}{\partial\xi}d\xi\right)^2 + \left(\frac{\partial z}{\partial\lambda}d\lambda + \frac{\partial z}{\partial\xi}d\xi\right)^2 \\ &= \left(\left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 + \left(\frac{\partial z}{\partial\lambda}\right)^2\right)d\lambda^2 + 2\left(\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\xi} + \frac{\partial y}{\partial\lambda}\frac{\partial y}{\partial\xi} + \frac{\partial z}{\partial\lambda}\frac{\partial z}{\partial\xi}\right)d\lambda d\xi \\ &\quad + \left(\left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 + \left(\frac{\partial z}{\partial\xi}\right)^2\right)d\xi^2 \\ &= g_{11}d\lambda^2 + (g_{12} + g_{21})d\lambda d\xi + g_{22}d\xi^2\end{aligned}$$

where $g_{12} = g_{21}$and all components g_{ij} are now functions of λ, ξ only. Knowing these components, we may compute the curvature by the methods described in the previous Note.