The Einstein Equation

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From the Riemann curvature tensor, we can construct simpler objects. The trace gives the Ricci tensor

$$R_{\alpha\beta} = R^{\mu}_{\ \alpha\mu\beta}$$

and the trace of this gives the Ricci scalar,

$$R = g^{\mu\nu} R_{\mu\nu}$$

As we showed in class, traces of the Bianchi identities

$$\begin{aligned} R^{\mu}_{\ [\alpha\nu\beta]} &= 0 \\ R^{\mu}_{\ \alpha[\nu\beta;\sigma]} &= 0 \end{aligned}$$

lead to two conclusions. The first shows that the Ricci tensor is symmetric, and the second shows that the combination

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$$

is divergence free:

$$G^{\alpha\beta}_{\quad ;\beta} = 0$$

The combination $G^{\alpha\beta}$ is called the Einstein tensor. It is both symmetric and divergence free. These are also the properties we demand of the stress-energy tensor. Briefly, the stress energy tensor contains all information about the energy content of matter, including energy density, momentum flux, pressures and material stresses. Since the Einstein tensor is the only tensor linear in components of the curvature that has these to properties, it is natural to set

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}$$

This is the Einstein equation. It is a non-linear, second-order differential equation for the metric with matter as the source. Given the matter content, $T_{\alpha\beta}$ of a region of spacetime, we may (try to) solve this equation for $g_{\alpha\beta}$. Knowing the metric, we may then compute the entire curvature tensor and connection. This tells us whatever we need to know about the geometry. In particular we may look at geodesics to find the paths followed by the gravitational influence of $T_{\alpha\beta}$.

The most important solutions of the Einstein equation are for systems that are nearly spherical. Such solutions describe stars and planets. The simplest such solution is the Schwarzschild solution for a static, spherically symmetric spacetime. This corresponds to a non-rotating planet or star, and allows us to make comparisons with Newtonian gravity. While rotating solutions are more realistic, many novel features can be seen from the Schwarzschild solution.

The metric of the most general static, spherically symmetric spacetime may be put in the form

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

and you have found the curvatures associated with this entire class of metrics. Now we want to know which of these geometries, if any, solve the Einstein equation. We are interested in the case when $T_{\alpha\beta} = 0$ because this describes the spacetime *outside* the surface of the planet or star – the empty space near a gravitating body. Therefore, consider

$$G_{\alpha\beta} = 0$$

It turns out that for "vacuum" solutions (i.e., those with $T_{\alpha\beta} = 0$), there is a simpler equation. Notice that if we take the trace of the vacuum Einstein equation,

$$\begin{aligned} 0 &= g^{\alpha\beta} G_{\alpha\beta} \\ &= g^{\alpha\beta}\left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R\right) \\ &= R - \frac{1}{2}\cdot 4R \\ &= -R \end{aligned}$$

the Ricci scalar must be zero. Putting this back into the Einstein tensor, it implies

$$R_{\alpha\beta} = 0$$

Therefore, it is not necessary to find the Einstein tensor to solve for vacuum spacetimes. It is equivalent to set the Ricci tensor to zero.

Problem 1: Starting from your solution for the components of the Riemann curvature tensor, compute all 10 components (it is symmetric!) of the Ricci tensor. For example, write out the sum for the tt component,

$$\begin{aligned} R_{tt} &= R^{\alpha}{}_{t\alpha t} \\ &= R^{t}{}_{ttt} + R^{r}{}_{trt} + R^{\theta}{}_{t\theta t} + R^{\varphi}{}_{t\varphi t} \end{aligned}$$

then substitute your solution for the components of curvature that you need. I recommend writing out the terms you'll need this way for each component of the Ricci tensor. It makes it easier to see which bits you need without losing any. Make a list of the nonzero components of the Ricci tensor.

Problem 2: Set each component of the Ricci tensor to zero. This should give you four equations that depend on f, g and their first and second derivatives. Two of these equations will be equivalent, and a third may be found by differentiating the others. Therefore, you will have two independent, coupled, ordinary differential equations for the two functions f and g. Solve them.