Cosmology

April 13, 2015

1 The cosmological principle

Cosmology is based on the principle that on large scales, space (not spacetime) is homogeneous and isotropic – that there is no preferred location or direction in the cosmos. This is strongly supported by the data. The principle means that we may write the spatial part of the spacetime metric as a time-dependent multiple of a constant-curvature space. We have found the curvature tensor for such maximally symmetric spaces. Combining this with a time-dependent factor, $a(t)$, and the time part of the metric, the line element becomes

$$\begin{aligned} ds^2 &= -dt^2 + a^2\left(\delta_{ij} + \frac{\kappa}{1-\kappa r^2}x_i x_j\right)dx^i dx^j \\ & -dt^2 + a^2 h_{ij}dx^i dx^j \end{aligned}$$

where $\kappa = \pm 1$, $i, j = 1, 2, 3$, and $a = a(t)$ sets the cosmic distance scale at any given time and h_{ij} is our constant curvature 3-metric.

The connection is easily found,

$$\begin{aligned} -\Gamma_{0ij} &= \Gamma_{i0j} = \Gamma_{ij0} = a\dot{a}h_{ij} \\ \Gamma_{ijk} &= a^2\tilde{\Gamma}_{ijk} \end{aligned}$$

where $\tilde{\Gamma}_{ijk}$ is the connection arising purely from the maximally symmetric metric, h_{ij}. Therefore,

$$\begin{aligned} \Gamma^i_{0j} &= \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^0_{ij} &= a\dot{a}h_{ij} \\ \Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} \end{aligned}$$

The curvature can also be written in terms of maximally symmetric parts, and parts depending on $a(t)$.

$$\begin{aligned} R^i_{jkl} &= \Gamma^i_{jl,k} - \Gamma^i_{jk,l} - \Gamma^i_{bl}\Gamma^b_{jk} + \Gamma^i_{bk}\Gamma^b_{jl} \\ &= \tilde{R}^i_{jkl} - \Gamma^i_{0l}\Gamma^0_{jk} + \Gamma^i_{0k}\Gamma^0_{jl} \\ &= \tilde{R}^i_{jkl} + \dot{a}^2\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right) \end{aligned}$$

and replacing $\tilde{R}^i_{jkl}$ with the expression for the maximally symmetric curvature,

$$R^i_{jkl} = \left(\kappa + \dot{a}^2\right)\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right)$$

Next, consider

$$R^0_{jkl} = \Gamma^0_{jl,k} - \Gamma^0_{jk,l} - \Gamma^0_{bl}\Gamma^b_{jk} + \Gamma^0_{bk}\Gamma^b_{jl}$$

This must be proportional to some rank-3 tensor in the maximally symmetric space, but there is none so we expect these components to vanish. Indeed, we find

$$\begin{aligned}
R^0_{jkl} &= a\dot{a}h_{jl,k} - a\dot{a}h_{jk,l} - a\dot{a}h_{ml}\tilde{\Gamma}^m_{jk} + a\dot{a}h_{ml}\tilde{\Gamma}^m_{jk} \\
&= a\dot{a}\left(h_{jl,k} - h_{ml}\tilde{\Gamma}^m_{jk} - h_{jm}\tilde{\Gamma}^m_{lk} - h_{jk,l} + h_{ml}\tilde{\Gamma}^m_{jk} + h_{jm}\tilde{\Gamma}^m_{kl}\right) \\
&\quad + h_{jm}\tilde{\Gamma}^m_{lk} - h_{jm}\tilde{\Gamma}^m_{kl} \\
&= a\dot{a}\left(h_{jl;k} - h_{jk;l}\right) \\
&= 0
\end{aligned}$$

where the derivatives of h_{ij} in the penultimate step are with respect to the maximally symmetric connection. Since h_{ij} is the metric compatible with this connection, the derivatives vanish.

The final components are

$$\begin{aligned}
R^0_{j0l} &= \Gamma^0_{jl,0} - \Gamma^0_{j0,l} - \Gamma^0_{bl}\Gamma^b_{j0} + \Gamma^0_{b0}\Gamma^b_{jl} \\
&= \Gamma^0_{jl,0} - \Gamma^0_{bl}\Gamma^b_{j0} \\
&= \left(a\ddot{a} + \dot{a}^2\right)h_{jl} - a\dot{a}h_{ml}\frac{\dot{a}}{a}\delta^m_j \\
&= a\ddot{a}h_{jl}
\end{aligned}$$

Collecting terms, we have

$$\begin{aligned}
R^0_{j0l} &= a\ddot{a}h_{jl} \\
R^i_{jkl} &= \left(\kappa + \dot{a}^2\right)\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right)
\end{aligned}$$

and terms related to these by symmetry.

The Ricci tensor follows immediately,

$$\begin{aligned}
R_{00} &= R^i_{0i0} \\
&= -\frac{1}{a^2}h^{ij}R^0_{j0i} \\
&= -\frac{1}{a^2}h^{ij}a\ddot{a}h_{ji} \\
&= -\frac{3\ddot{a}}{a}
\end{aligned}$$

and

$$\begin{aligned}
R_{ij} &= R^0_{i0j} + R^m_{imj} \\
&= a\ddot{a}h_{ij} + \left(\kappa + \dot{a}^2\right)\left(\delta^m_m h_{ij} - \delta^m_i h_{mj}\right) \\
&= \left(a\ddot{a} + 2\kappa + 2\dot{a}^2\right)h_{ij}
\end{aligned}$$

and the Ricci scalar is

$$\begin{aligned}
R &= g^{00}R_{00} + \frac{1}{a^2}h^{ij}R_{ij} \\
&= \frac{3\ddot{a}}{a} + \frac{3}{a^2}\left(a\ddot{a} + 2\kappa + 2\dot{a}^2\right) \\
&= \frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right)
\end{aligned}$$

Finally, we have the components of the Einstein tensor,

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2} g_{00} R \\ &= -\frac{3\ddot{a}}{a} + \frac{3\ddot{a}}{a} + \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) \\ &= \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) \end{aligned}$$

and

$$\begin{aligned} G_{ij} &= R_i - \frac{1}{2} g_{ij} R \\ &= \left(a\ddot{a} + 2\kappa + 2\dot{a}^2\right) h_{ij} - \frac{1}{2} a^2 h_{ij} \left(\frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right)\right) \\ &= \left(a\ddot{a} + 2\kappa + 2\dot{a}^2 - 3a\ddot{a} - 3\left(\kappa + \dot{a}^2\right)\right) h_{ij} \\ &= -\left(2a\ddot{a} + \kappa + \dot{a}^2\right) h_{ij} \end{aligned}$$

2 The energy tensor

Now consider possible energy tensors to serve as a source. At the current epoch, there is very little pressure and we may take the energy tensor for pressureless dust,

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

with its conservation equation,

$$\begin{aligned} 0 &= T^{\mu\nu}{}_{;\nu} \\ &= T^{\mu\nu}{}_{,\nu} + T^{\beta\nu}\Gamma^{\mu}{}_{\beta\nu} + T^{\mu\beta}\Gamma^{\nu}{}_{\beta\nu} \end{aligned}$$

The vanishing divergence is an identity for all but the time component, which becomes

$$\begin{aligned} 0 &= T^{0\nu}{}_{,\nu} + T^{\beta\nu}\Gamma^{0}{}_{\beta\nu} + T^{0\beta}\Gamma^{\nu}{}_{\beta\nu} \\ &= T^{00}{}_{,0} + T^{00}\Gamma^{0}{}_{00} + T^{00}\Gamma^{\nu}{}_{0\nu} \\ &= \dot{\rho} + T^{00}\Gamma^{i}{}_{0i} \\ &= \dot{\rho} + \rho\frac{3\dot{a}}{a} \end{aligned}$$

Multiplying by a^3 we have

$$\begin{aligned} 0 &= a^3\dot{\rho} + 3a^2\dot{a}\rho \\ &= \frac{d}{dt}\left(\rho a^3\right) \end{aligned}$$

so that ρa^3 remains constant.

The meaning of this is easy to see. Consider a unit spatial volume, $V_0 = l_0^3 = 1$. The comoving lengths of the sides will expand with a factor of a,

$$l(t) = a(t)$$

so that the volume at any time is given by

$$V(t) = a^3$$

The mass contained in this volume is

$$m \equiv \rho V = \rho a^3 = constant$$

3 From the Einstein equation to the Friedmann equation

We may now write the Einstein equation, including a possible cosmological constant, Λ,

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \beta T_{\alpha\beta}$$

where $\beta = \frac{8\pi G}{c^4}$. Each term in the equation is diagonal.

For the 00 component,

$$\begin{aligned} G_{00} + \Lambda g_{00} &= \beta T_{00} \\ \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) - \Lambda &= \frac{\beta m}{a^3} \end{aligned}$$

while the spatial components give a second equation,

$$\begin{aligned} G_{ij} + \Lambda g_{ij} &= 0 \\ -\left(2a\ddot{a} + \kappa + \dot{a}^2\right) h_{ij} + \Lambda a^2 h_{ij} &= 0 \end{aligned}$$

Notice that if we multiply the 00 equation by a^2 to get

$$3\left(\kappa + \dot{a}^2\right) - \Lambda a^2 = \frac{\beta m}{a}$$

then differentiate with respect to time,

$$\begin{aligned} 6\dot{a}\ddot{a} - 2\Lambda a\dot{a} &= -\frac{\beta m}{a^2}\dot{a} \\ &= -\frac{1}{a}\left(3\left(\kappa + \dot{a}^2\right) - \Lambda a^2\right)\dot{a} \end{aligned}$$

where we substitute the original expression for $\frac{\beta m}{a}$. Cancelling a common factor of $\dot{a}$ and simplifying,

$$\begin{aligned} 6\ddot{a} - 3\Lambda a &= -\frac{3}{a}\left(\kappa + \dot{a}^2\right) \\ 2a\ddot{a} - \Lambda a^2 &= -\left(\kappa + \dot{a}^2\right) \end{aligned}$$

so finally,

$$2a\ddot{a} - \Lambda a^2 + \kappa + \dot{a}^2 = 0$$

and we have derived the spatial equation from the 00 component. This is consistent with the requirements of the Bianchi identity.

We have therefore reduced this cosmological model to the single, first-order *Friedmann equation*,

$$3a\left(\kappa + \dot{a}^2\right) - \Lambda a^3 - \beta m = 0$$

Solutions to this govern low energy cosmology. At early times, when the energies and pressures are substantial, this requires modification.

4 Curvature singularity

The metric appears to be degenerate if $a = 0$ or if a diverges. The first of these is also a curvature singularity. We may confirm this by substituting the field equations into the expression for the scalar curvature,

$$R = \frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right)$$

To evaluate this, start with the field equations,

$$\begin{aligned} 2a\ddot{a} + \kappa + \dot{a}^2 - \Lambda a^2 &= 0 \\ \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) - \Lambda - \frac{\beta m}{a^3} &= 0 \end{aligned}$$

Solving the first for $\ddot{a}$ shows that

$$\begin{aligned} \frac{6\ddot{a}}{a} &= \frac{6}{a}\left(-\frac{\kappa + \dot{a}^2 - \Lambda a^2}{2a}\right) \\ &= -\frac{3}{a^2}\left(\kappa + \dot{a}^2\right) + 3\Lambda \end{aligned}$$

while the second immediately gives

$$\frac{3}{a^2}\left(\kappa + \dot{a}^2\right) = \Lambda + \frac{\beta m}{a^3}$$

Combining in the expression for R,

$$\begin{aligned} R &= \frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right) \\ &= -\frac{3}{a^2}\left(\kappa + \dot{a}^2\right) + 3\Lambda + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right) \\ &= 3\Lambda + \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) \\ &= 4\Lambda + \frac{\beta m}{a^3} \end{aligned}$$

which diverges when $a = 0$.

It is not hard to show that other curvature invariants also diverge at $a = 0$, and nowhere else.

Properties of the Friedmann equation

We now examine the Friedmann equation,

$$3a\left(\kappa + \dot{a}^2\right) - \Lambda a^3 - \beta m = 0$$

Solving for $\dot{a}$,

$$\dot{a} = \pm\sqrt{\frac{\Lambda a^3 + \beta m - 3\kappa a}{3a}}$$

The rate of change of a is therefore divergent at $a = 0$, and when a diverges. The exact solution is given by integrating

$$\pm\left(t - t_0\right) = \int_{a(t_0)}^{a(t)} \frac{\sqrt{3a}\,da}{\sqrt{\Lambda a^3 + \beta m - 3\kappa a}}$$

but this is not necessary to see the qualitative properties of the possible solutions. Instead, for various signs of Λ and κ, we treat the problem as a single particle in a potential. We need to consider 4 cases, depending on the signs of Λ and κ. We may always take $a > 0$.

Case 1: Positive spatial curvature and positive cosmological constant

When $\kappa = 1$ and $\Lambda > 0$,

$$\dot{a} = \pm\sqrt{\frac{1}{3}\Lambda a^2 + \frac{\beta m}{3a} - 1}$$

If we write the Friedmann equation as

$$-1 = -\kappa = \dot{a}^2 - \frac{1}{3}\Lambda a^2 - \frac{\beta m}{3a}$$

we have conservation of the "energy", $-\kappa$. The effective potential is strongly attractive at all values of a:

$$V_{eff} = -\frac{1}{3}\Lambda a^2 - \frac{\beta m}{3a}$$

with a single maximum at

$$\begin{aligned} 0 &= \frac{dV_{eff}}{da} \\ &= -\frac{2}{3}\Lambda a + \frac{\beta m}{3a^2} \\ a_{crit} &= \left(\frac{\beta m}{2\Lambda}\right)^{1/3} \end{aligned}$$

(There are three roots: $\left(\frac{\beta m}{2\Lambda}\right)^{1/3}$, $\left(\frac{\beta m}{2\Lambda}\right)^{1/3} e^{\frac{2\pi i}{3}}$, $\left(\frac{\beta m}{2\Lambda}\right)^{1/3} e^{\frac{4\pi i}{3}}$ only one of which is real). The value of the potential at this maximum is

$$\begin{aligned} V_{eff}\left(a_{crit}\right) &= -\frac{1}{3}\Lambda\left(\frac{\beta m}{2\Lambda}\right)^{2/3} - \frac{\beta m}{3}\left(\frac{\beta m}{2\Lambda}\right)^{-1/3} \\ &= -\frac{1}{3}\left(\frac{\beta m}{2\Lambda}\right)^{-1/3}\left(\Lambda\left(\frac{\beta m}{2\Lambda}\right) + \beta m\right) \\ &= -\left(\left(\frac{\beta m}{2}\right)^3 \frac{2\Lambda}{\beta m}\right)^{1/3} \\ &= -\left(\frac{1}{4}\beta^2 m^2 \Lambda\right)^{1/3} \end{aligned}$$

The evolution of the system depends on how this value compares to the total energy, $-\kappa = -1$. If

$$\frac{1}{4}\beta^2 m^2 \Lambda < 1$$

then the energy exceeds the maximum of the potential and a increases or decreases monotonically to infinity or zero, depending on the sign of its initial condition, $\dot{a}\left(t_0\right)$. On the other hand, if

$$\frac{1}{4}\beta^2 m^2 \Lambda > 1$$

there is a turning point when $V_{eff} = -1$,

$$\begin{aligned} -1 &= -\frac{1}{3}\Lambda a_{turning}^2 - \frac{\beta m}{3a_{turning}} \\ 0 &= \frac{1}{3}\Lambda a_{turning}^3 - a_{turning} + \frac{\beta m}{3} - 1 \end{aligned}$$

If the initial value, $a(t_0)$, is below a_{crit}, the universe expands to this turning point, then contracts to the singularity. If $a(t_0)$ exceeds a_{crit}, then the universe contracts to the turning point then expands forever.

The complete solution is given by integrating,

$$\begin{aligned} \pm\int_{t_0}^{t} dt &= \int_{a(t_0)}^{a} \frac{da}{\sqrt{\frac{1}{3}\Lambda a^2 + \frac{\beta m}{3a} - 1}} \\ \pm(t - t_0) &= \int_{a(t_0)}^{a} \frac{\sqrt{3a}da}{\sqrt{\Lambda a^3 - 3a + \beta m}} \end{aligned}$$

Case 2: Positive spatial curvature and negative cosmological constant

When $\kappa = 1$ and $\Lambda < 0$, the energy equation takes the form

$$-1 = \dot{a}^2 + \frac{1}{3}|\Lambda| a^2 - \frac{\beta m}{3a}$$

and the effective potential is

$$V_{eff} = \frac{1}{3}|\Lambda| a^2 - \frac{\beta m}{3a}$$

This has no extrema,

$$\begin{aligned} 0 &= \frac{dV_{eff}}{da} \\ &= \frac{2}{3}|\Lambda| a + \frac{\beta m}{3a^2} \end{aligned}$$

since

$$a^3 = -\frac{\beta m}{2|\Lambda|}$$

has only one negative and two complex roots. The potential becomes infinitely attractive for a small, and increases without bound for large a, with a single inflection point at

$$\begin{aligned} 0 &= \frac{d^2V_{eff}}{da^2} \\ &= \frac{2}{3}|\Lambda| - \frac{2\beta m}{3a^3} \\ a &= \left(\frac{\beta m}{|\Lambda|}\right)^{1/3} \end{aligned}$$

The universe expands to a maximum at the single real positive root of

$$0 = \frac{1}{3}|\Lambda| a^3 + a - \frac{\beta m}{3}$$

then collapses to $a = 0$.

Case 3: Negative spatial curvature and positive cosmological constant

When $\kappa = -1$ and the cosmological constant is positive, we have

$$1 = \dot{a}^2 - \frac{1}{3}\Lambda a^2 - \frac{\beta m}{3a}$$

which again has negative definite potential with a single maximum at

$$V_{eff}\left(a_{crit}\right) = -\left(\frac{1}{4}\beta^2 m^2 \Lambda\right)^{1/3}$$

Since the energy is now positive, $\dot{a}^2$ remains nonzero and there is no turning point. The universe continues to collapse or expand in accordance with the initial sign of $\dot{a}$.

Case 4: Negative spatial curvature and negative cosmological constant

When $\kappa = -1$ and the cosmological constant is positive,

$$1 = \dot{a}^2 + \frac{1}{3}\left|\Lambda\right| a^2 - \frac{\beta m}{3a}$$

so we again have the monotonic potential,

$$V_{eff} = \frac{1}{3}\left|\Lambda\right| a^2 - \frac{\beta m}{3a}$$

This always eventually exceeds the energy, so we have a single turning point when

$$0 = \frac{1}{3}\left|\Lambda\right| a^3 - a - \frac{\beta m}{3}$$