Examples of Tensors

February 3, 2013

We will develop a number of tensors as we progress, but there are a few that we can describe immediately. We look at two cases: (1) the spacetime tensor description of electromagnetism, and (2) energy-momentum tensors.

1 Electromagnetism in special relativity

A complete treatment of this topic is readily available on wikipedia. The link is given in the Notes.

The 4-vector potential is built from the magnetic vector potential $\mathbf{A}$ and the electric potential φ as

$$A^\alpha = \left(\frac{\varphi}{c}, \mathbf{A}\right)$$

where the electric and magnetic fields are given by

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A} \end{aligned}$$

In order for the spatial components to arise from the 4-dimensional description, we consider the magnetic field in components:

$$\begin{aligned} B^i &= \varepsilon^{ij}{}_k \frac{\partial}{\partial x^j} A^k \\ &= \varepsilon^{ijk} \partial_j A_k \end{aligned}$$

There is no 4-dimensional equivalent of the cross-product, because the 4-dimensional Levi Civita tensor, $\varepsilon_{\alpha\beta\mu\nu}$, cannot turn the derivatives of a vector $\partial_\alpha A_\beta$, into another vector. Nonetheless, we can still write $\partial_\alpha A_\beta - \partial_\beta A_\alpha$. In three dimensions, we can also write this by inverting the ε^{ijk} above:

$$\begin{aligned} B^i &= \varepsilon^{ijk} \partial_j A_k \\ \varepsilon_{imn} B^i &= \varepsilon_{imn} \varepsilon^{ijk} \partial_j A_k \\ B^i \varepsilon_{imn} &= \left(\delta^j_m \delta^k_n - \delta^k_m \delta^j_n\right) \partial_j A_k \\ &= \partial_m A_n - \partial_n A_m \end{aligned}$$

This is the clue we need.

Define the Faraday tensor,

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

Then the spatial components, F_{ij}, become

$$\begin{aligned} F_{jk} &= \partial_j A_k - \partial_k A_j \\ &= B^i \varepsilon_{ijk} \\ &= \begin{pmatrix} 0 & B^3 & -B^2 \\ -B^3 & 0 & B^1 \\ B^2 & -B^1 & 0 \end{pmatrix} \end{aligned}$$

For the remaining components, set $\alpha = 0$. Then, noticing that

$$\begin{aligned} A_\alpha &= \eta_{\alpha\beta} A^\beta \\ &= \left(-\frac{\varphi}{c}, \mathbf{A}\right) \end{aligned}$$

we have

$$\begin{aligned} F_{0\beta} &= \partial_0 A_\beta - \partial_\beta A_0 \\ F_{00} &= \partial_0 A_0 - \partial_0 A_0 = 0 \\ F_{0i} &= \partial_0 A_i - \partial_i (-\varphi) \\ &= \frac{1}{c}\frac{\partial}{\partial t} A_i - \partial_i \left(-\frac{\varphi}{c}\right) \\ &= -E_i \end{aligned}$$

and, from the antisymmetry, $F_{i0} = E_i = \delta_{ij} E^j$, showing that the electric and magnetic fields are both parts of a single rank-2 tensor,

$$\begin{aligned} F_{\alpha\beta} &= \begin{pmatrix} 0 & -\frac{E^1}{c} & -\frac{E^2}{c} & -\frac{E^3}{c} \\ \frac{E^1}{c} & 0 & B^3 & -B^2 \\ \frac{E^2}{c} & -B^3 & 0 & B^1 \\ \frac{E^3}{c} & B^2 & -B^1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \end{aligned}$$

where in the second form we have set $c = 1$.

We may now write Maxwell's equations in terms of these. Four equations follow as identities. Since we have

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

it follows that taking a derivative and antisymmetrizing on all three indices gives zero:

$$\partial_\mu F_{\alpha\beta} = \partial_\mu \partial_\alpha A_\beta - \partial_\mu \partial_\beta A_\alpha$$

To antisymmetrize, notice that

$$\begin{aligned} \partial_{[\mu} F_{\alpha\beta]} &\equiv \frac{1}{3!}\left(\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} - \partial_\mu F_{\beta\alpha} - \partial_\alpha F_{\mu\beta} - \partial_\beta F_{\alpha\mu}\right) \\ &= \frac{1}{3!}\left(\left(\partial_\mu F_{\alpha\beta} - \partial_\mu F_{\beta\alpha}\right) + \left(\partial_\alpha F_{\beta\mu} - \partial_\alpha F_{\mu\beta}\right) + \left(\partial_\beta F_{\mu\alpha} - \partial_\beta F_{\alpha\mu}\right)\right) \\ &= \frac{1}{3}\left(\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha}\right) \end{aligned}$$

This simplification always occurs: when antisymmetrizing an expression, the number of terms is reduced if part of the expression (in this case, $F_{\alpha\beta} = -F_{\beta\alpha}$) is already antisymmetric. Writing this in terms of the potential,

$$\begin{aligned} \partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} &= \partial_\mu \partial_\alpha A_\beta - \partial_\mu \partial_\beta A_\alpha + \partial_\alpha \partial_\beta A_\mu - \partial_\alpha \partial_\mu A_\beta + \partial_\beta \partial_\mu A_\alpha - \partial_\beta \partial_\alpha A_\mu \\ &= \left(\partial_\mu \partial_\alpha A_\beta - \partial_\alpha \partial_\mu A_\beta\right) + \left(\partial_\beta \partial_\mu A_\alpha - \partial_\mu \partial_\beta A_\alpha\right) + \left(\partial_\alpha \partial_\beta A_\mu - \partial_\beta \partial_\alpha A_\mu\right) \\ &= 0 \end{aligned}$$

because partial derivatives commute, $\frac{\partial^2 A_\beta}{\partial x^\mu \partial x^\alpha} = \frac{\partial^2 A_\beta}{\partial x^\alpha \partial x^\mu}$. Therefore, we have

$$\partial_{[\mu} F_{\alpha\beta]} = 0$$

This gives two of the Maxwell equations. To see this, first notice that because of the antisymmetry, μ, α, β must all be different. This gives two cases. First let $\alpha = 0, \beta = i, \mu = j$ and write out the components

$$\begin{aligned} 0 &= \partial_j F_{0i} + \partial_0 F_{ij} + \partial_i F_{j0} \\ &= -\partial_j E_i + \partial_0 B^m \varepsilon_{ijm} + \partial_i E_j \end{aligned}$$

Contracting with the 3-dimensional Levi-Civita tensor,

$$\begin{aligned} 0 &= \varepsilon^{ijk} \left(\partial_i E_j - \partial_j E_i \right) + \partial_0 B^m \varepsilon^{ijk} \varepsilon_{ijm} \\ &= 2\varepsilon^{ijk} \partial_i E_j + \partial_0 B^m \left(2\delta^k_m \right) \\ &= 2 \left[\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right]^k \end{aligned}$$

The second case is when α, β, μ are all spatial,

$$\begin{aligned} 0 &= \partial_j F_{ki} + \partial_k F_{ij} + \partial_i F_{jk} \\ &= \partial_j \left(B^m \varepsilon_{mki} \right) + \partial_k \left(B^m \varepsilon_{mij} \right) + \partial_i \left(B^m \varepsilon_{mjk} \right) \end{aligned}$$

Contract this with ε^{ijk},

$$\begin{aligned} 0 &= \varepsilon^{ijk} \varepsilon_{mki} \partial_j B^m + \varepsilon^{ijk} \varepsilon_{mij} \partial_k B^m + \varepsilon^{ijk} \varepsilon_{mjk} \partial_i B^m \\ 0 &= 2\delta^j_m \partial_j B^m + 2\delta^k_m \partial_k B^m + 2\delta^i_m \partial_i B^m \\ 0 &= 2 \left(\partial_m B^m + \partial_m B^m + \partial_m B^m \right) \\ 0 &= 6 \left(\boldsymbol{\nabla} \cdot \mathbf{B} \right) \end{aligned}$$

We now have the two homogeneous Maxwell equations,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}$$

written as a single 4-dimensional equation, $\partial_{[\mu} F_{\alpha\beta]} = 0$.

For the remaining Maxwell equations, notice that we need the divergence of the electric field, $\boldsymbol{\nabla} \cdot \mathbf{E}$. Since

$$\begin{aligned} F^{\alpha\beta} &= \eta^{\alpha\mu} F_{\mu\nu} \eta^{\nu\beta} \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \end{aligned}$$

we can write

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \partial_i F^{0i}$$

so we guess that the object to consider is

$$\partial_\beta F^{\alpha\beta}$$

Then the time component, $\alpha = 0$, gives the divergence of $\mathbf{E}$, while the spatial components, $\alpha = i$, become

$$\begin{aligned}
\partial_\beta F^{i\beta} &= \partial_0 F^{i0} + \partial_j F^{ij} \\
&= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}E^i\right) + \frac{\partial}{\partial x^j}\varepsilon^{ij}{}_k B^k \\
&= -\frac{1}{c^2}\frac{\partial E^i}{\partial t} + \varepsilon^{ij}{}_k \frac{\partial}{\partial x^j}B^k \\
&= -\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B}
\end{aligned}$$

These give the inhomogeneous Maxwell equations if we define the 4-current,

$$J^\alpha = (\rho c, \mathbf{J})$$

so that

$$\partial_\beta F^{\alpha\beta} = \frac{4\pi}{c}J^\alpha$$

is equivalent to

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\rho \\
\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}
\end{aligned}$$

For the energy content of the electromagnetic field, see below.

2 Energy-momentum tensor

The energy-momentum tensor describes the energy content of a region of spacetime. Its non-relativistic spatial part is the stress tensor from classical mechanics. Read in Schutz for more detail on these components. The T^{00} component is the energy density $\frac{kg\,c^2}{m^3}$, while the T^{0i} components give momentum flux density $\left(\frac{kg\,m}{sec}\times\frac{1}{m^2}\times\frac{1}{sec}\right)$.

2.1 Electrodynamic energy-momentum

For the energy-momentum content of the electromagnetic field, we first consult Jackson, Section 6.7. Here we find the energy density of the electromagnetic field in vacuum eq. 6.106,

$$u = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)$$

and the momentum density (proportional to the Poynting vector) eq. 6.118,

$$\mathbf{g} = \frac{1}{c^2}\mathbf{E}\times\mathbf{B}$$

while the spatial components of the Maxwell stress tensor take the form

$$\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{E}\right) - \mathbf{E}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right)$$

These are the parts of the full energy-momentum tensor,

$$T^{\alpha\beta} = F^{\alpha\mu}F^{\beta}_{\ \mu} - \frac{1}{4}\eta^{\alpha\beta}\left(F^{\mu\nu}F_{\mu\nu}\right)$$

2.2 The energy-momentum tensor of a scalar field

For other field theories, there are similar expressions. For example, a Klein-Gordon (massive scalar) field satisfies

$$-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}+\nabla^2\varphi=\frac{m^2c^2}{\hbar^2}\varphi$$

The non-relativistic limit of this equation is the Schrödinger equation. The energy-momentum tensor for this field is given by

$$T^{\alpha\beta}=\partial^\alpha\varphi\partial^\beta\varphi-\frac{1}{4}\eta^{\alpha\beta}\left(\partial^\mu\varphi\partial_\mu\varphi\right)$$

where

$$\partial^\alpha=\eta^{\alpha\beta}\partial_\beta$$

2.3 The energy-momentum tensor of a perfect fluid

For cosmological models, we approximate the distribution of matter in universe as a fluid. A perfect fluid is one with no viscosity or heat flow, and show in Chapter 4 or Schutz to have an energy-momentum tensor of form

$$T^{\alpha\beta}=\begin{pmatrix}\rho&&&\\&p&&\\&&p&\\&&&p\end{pmatrix}$$

where the pressure p is spatially isotropic and the density is ρ. We may write this tensor in terms of the local 4-velocity $u^\alpha=(c,0,0,0)$ of the matter and the metric. We can get the pressure pieces using the metric,

$$p\eta^{\alpha\beta}=\begin{pmatrix}-p&0&0&0\\0&p&0&0\\0&0&p&0\\0&0&0&p\end{pmatrix}$$

but this gives the wrong expression for T^{00}. We fix this by adding

$$\left(\rho+\frac{p}{c^2}\right)u^\alpha u^\beta=\begin{pmatrix}\left(\rho+\frac{p}{c^2}\right)c^2&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

Checking units, we see that ρc^2 and p both have units of energy density:

$$\begin{aligned}\left[\frac{p}{c^2}\right]&=\frac{F}{Ac^2}\\&=\frac{kg\,m}{s^2\,m^2}\frac{s^2}{m^2}\\&=\frac{kg}{m^3}\\ [\rho]&=\frac{kg}{m^3}\end{aligned}$$

Adding the two expressions, we have

$$\begin{aligned}T^{\alpha\beta}&=\left(\rho+\frac{p}{c^2}\right)u^\alpha u^\beta+p\eta^{\alpha\beta}\\&=\left(\rho+p\right)u^\alpha u^\beta+p\eta^{\alpha\beta}\end{aligned}$$

2.4 General properties of the energy-momentum tensor

The energy momentum tensor for any matter will have two essential properties. The first is symmetry,

$$T^{\alpha\beta} = T^{\beta\alpha}$$

This follows from conservation of angular momentum, guaranteeing that a small volume element of the material will not spontaneously start to rotate.

The second property is the conservation of energy and momentum, which may always be expressed as the vanishing of the divergence of the tensor,

$$\partial_\alpha T^{\alpha\beta} = 0$$

To see how this implies conservation of energy, we integrate the $\beta = 0$ component of the 3-divergence over an arbitrary spatial volume, V^3, and use the divergence theorem,

$$\int\limits_{V^3} \partial_i T^{i0} d^3x = \int\limits_{S^2=\delta V^3} n_i T^{i0} d^2x$$

where n^i is the outward normal to the 2-dimensional spatial boundary of V^3. That integral gives the energy flowing out across the boundary surface. When the 4-divergence of $T^{\alpha\beta}$ vanishes, we have

$$\partial_i T^{i\beta} = -\partial_0 T^{0\beta}$$

and therefore,

$$\begin{aligned}
\int\limits_{S^2=\delta V^3} n_i T^{i0} d^2x &= -\int\limits_{V^3} \partial_0 T^{00} d^3x \\
&= -\frac{d}{dt}\int\limits_{V^3} \rho c^2 d^3x \\
&= -\frac{dE}{dt}
\end{aligned}$$

where E is the total energy in the volume V^3. Therefore, the rate of change of energy in V^3 equals the negative of the rate at which energy flows across the boundary of V^3,

$$\frac{dE}{dt} = -\int\limits_{S^2=\delta V^3} n_i T^{i0} d^2x$$

Similarly, the rate of change of momentum in a volume V^3 is given by the spatial componenets, $\beta = k$,

$$\begin{aligned}
\int\limits_{V^3} \partial_0 T^{0k} d^3x &= -\int\limits_{V^3} \partial_i T^{ik} d^3x \\
\frac{d}{dt}\int\limits_{V^3} T^{0k} d^3x &= -\int\limits_{S^2} n_i T^{ik} d^2x
\end{aligned}$$

Again, the change in momentum of the volume is the integral of the momentum flux over the boundary.