Relativistic Stars

Reminder: Geometricized Units

• Just as a reminder, relativists work in *geometricized units*, where $G = c = 1$. As astrophysicists, this seems weird and we often endeavour to restore "proper units" to expressions we use and derived.

• There is some usefulness in using geometricized units because it allows you to *quickly* make assessments about whether or not you need to worry about relativistic considerations in any given astrophysical computation you may be engaged in.

• As a consideration of this, let's recall the *Schwarzschild radius*, r_s. While this physical quantity is associated with Karl Schwarzschild, who discovered the first exact solution of the Einstein Field Equations, it was actually *first* derived in *Newtonian* gravity!

• In 1783, the Rev. John Michell was thinking about the concept of escape speed, and asked how compact an object had to be in order for the escape speed to be the speed of light. This radius is sometimes referred to as the "gravitational radius", and the answer (J. Michell, *Phil. Trans. Roy. Soc. Lon.*, **74**, 35 [1783]) is

$$r_s = \frac{2GM}{c^2} \qquad \to \qquad r_s = 2M$$

• As you can see on the RHS of the "→" we have imposed geometricized units, and see the physical dependence directly — the Schwarzschild radius changes with the mass.

• The whole point in working with geometricized units is this: we have abandoned the idea that gravity is a field, instead replacing it with the idea that the "force of gravity" is a manifestation of the fact that *spacetime is curved*, and that observed motions ("under the influence of a gravitational field") are a consequence of the underlying geometry of spacetime.

• As such, geometricized units provide us with a convenient way to talk about whether or not physical quantities affect the *geometry of spacetime*. Relativistic astrophysicists will ask themselves, "is this quantity geometrically important?"

• The easy way to answer this is to directly compare physical quantities; in SI units, this is not always easy — how does the mass compare to the angular momentum? But in geometricized units, where everything has been reduced to units of *meters*, it becomes much easier!

• For reference, here is a table of the most common multiplicative conversion factors, which we will return to and use often in relativistic astrophysics.

Quantity	***Value***	***Notes***
G/c^2	7.426×10^{-28} m/kg	Convert mass to geometric units
G/c^3	2.477×10^{-36} s/kg	Convert angular momentum to geometric units
G/c^4	8.263×10^{-45} s/(kg m)	Convert energy to geometric units
G/c^5	2.756×10^{-53} $s^3/(kg\ m^2)$	Convert power (luminosity) to geometric units

• As astrophysicists are often used to talking about physical quantities in reference to known, easily measured values (like the values for the Sun), it is also convenient to carry around in your pocket a few useful numbers. There is no definitive set, but some that I use regularly are:

Symbol	***Value***	***Quantity***
$M_\odot$	1477 m	Solar mass
$L_\odot$	1.058×10^{-26}	Solar Luminosity (dimensionless)
$M_\oplus$	4.435×10^{-3} m	Earth mass

When does relativity matter for stars?

• At this point you have derived (or at least seen) the Schwarzschild solution to the Einstein Field Equations. Schwarzschild is the unique *static*, *spherically symmetric*, *vacuum* solution to the EFEs.

• In fact, the Schwarzschild solution is the ***only*** spherically symmetric solution to the vacuum Einstein Equations. This result is known as ***Birkhoff's Theorem***:

> Any *spherically symmetric, vacuum solution* of the EFEs must be *stationary* and *asymptotically flat*; this implies that the solution must be given by the *Schwarzschild metric.*

• An important consequence of this is that ***all*** spherical gravitational fields are indistinguishable from one another at large distances; we could not tell (for instance) if the Sun is a star, or a black hole of the same mass.

• In its classic form, the Schwarzschild metric may be written in spherical coordinates $\{t, r, \theta, \phi\}$ as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta \ d\phi^2$$

• The metric is entirely characterized in terms of a single parameter M, usually called *the mass*. Note that as $r \to \infty$, this becomes the Minkowski metric — the spacetime is *asymptotically flat*.

• If one goes through this same exercise solving the joint vacuum Einstein-Maxwell equations (the "electrovac" solution) you get a similar metric known as *Reissner-Nordstrom*:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - 2M/r + Q^2/r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$$

The Reissner-Nordstrom solution is exceedingly important in the study of black hole solutions, but is generally considered to not be astrophysically relevant since charged objects tend to attract opposite charges and discharge on very short timescales in space.

• Note that in Schwarzschild there is explicit time independence in the metric; there is no dependence on t, so there is a timelike Killing vector, $\xi^\alpha = \{1, 0, 0, 0\}$. There is also explicit spherical symmetry; the last two terms may be written as a two-sphere:

$$r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2 = r^2 d\Omega^2$$

Like t, there is no explicit dependence on the variable ϕ, so there is an associated Killing vector $\eta^\alpha = \{0, 0, 0, 1\}$. There are additionally two other spacelike Killing vectors, though they are messy to write in these coordinates.

• How do we know whether or not a star can (or should) be treated relativistically?

• Since the mass M is the only parameter, let's consider the limit where M is small. In this case, the line element can be expanded to yield (note we are really talking about M/r being small):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$$

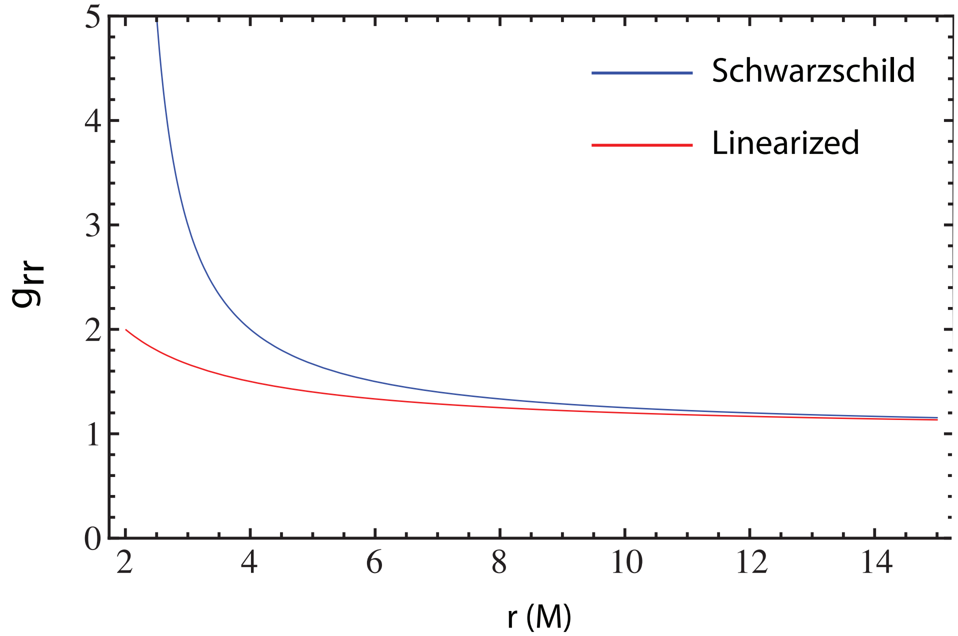
When studying *linearization* this is the exact form of the line element in the weak field limit where the Newtonian gravitational potential is given by

$$\Phi(r) = -\frac{M}{r} \qquad \to \qquad \Phi(r) = -\frac{GM}{r}$$

• A relativistic astrophysicist will ask "when is a physical quantity geometrically important?" As with all such questions in physics, the question is *up to you* as the observer; you must decide how much error you are willing to tolerate in your observations.

• In the above example, one can start by considering the mass, M. Does it matter? Or, in other words, can I treat a system as a *Newtonian* system, or do I have to use general relativity? Consider a plot comparing the g_{rr} component of the Schwarzschild metric compared to the linearized metric.

• Now let's consider a couple of different astrophysical systems, their parameters outlined in the table below:



System	***Mass***	***Radius***	$1 - g_{rr-linear}/g_{rr-Schw}$
Sun	$1M_\odot$	6.955×10^8 m $= 4.709 \times 10^5 M$	1.804×10^{-11}
White Dwarf	$1M_\odot$	6.371×10^6 m $= 4.314 \times 10^3 M$	2.149×10^{-7}
Sun	$1.4M_\odot$	10^4 m $= 7.143M$	0.0783969
Quark Star	$1M_\odot$	5000 m $= 3.38M$	0.350

• The last column in our table is the deviation between the linearized metric and the Schwarzschild metric (multiply this by 100 to get the percent difference); as general relativity becomes more important, this quantity grows because the linearized metric is becoming a poor approximation to the geometry of spacetime.

• ***ASIDE:*** You may notice in the graph that the value of g_{rr} for Schwarzschild is diverging as $r \to 2M$. This confused people for some time before it was realized that r is not the most ideal coordinate for all problems. This is a manifestation of a *coordinate singularity* — the coordinate itself is bad at $r = 2M$, while physical quantities are perfectly well behaved. Most of you are probably familiar with other coordinate singularities; the classic example is on the top of a 2-sphere (globe). What is the value of ϕ at the North Pole of the Earth? it is undefined because ϕ is a bad coordinate at that point, though there is nothing physically wrong at that point on the surface of the Earth, as can be proven by simply performing a rotation of the coordinate grid. *Caveat physitor!* You must always be on guard against interpreting weird coordinate effects as true physical conditions. Quantities that depend on coordinate values (such as tensor components) *are only valid for the observer who is attached to those coordinates!* To safeguard, consider *invariant* quantities whenever possible.

Particle Orbits in Schwarzschild

• As explorers, the way to understand a compact object and its effect on the space around it is to fly around and see what the geometry of the spacetime forces you to do! As astronomers, we do this all the time, watching astrophysical particles (gas, dust, comets, spaceships) orbit around compact objects.

• In the case of Schwarzschild, the highly symmetric geometry makes our life easy, because of the existence of the Killing vectors, two of which we mentioned are the timelike Killing vector ξ^α associated with the time independence, and the spacelike Killing vector η^α associated with the rotational ϕ symmetry.

• Noether's theorem tells us that for every symmetry there is a conserved quantity, and Killing vectors are a powerful way to find those conserved quantities. In particular, along *geodesics*, there is a conserved quantity along the geodesic given by

$$const = \zeta^\alpha u_\alpha$$

where ζ^α is any Killing vector, and u^α is the 4-velocity of an observer falling along the geodesic.

• We find it useful to define two conserved quantities associated with the two afore-mentioned Killing vectors:

$$\epsilon = -\xi^\alpha u_\alpha = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

and

$$\ell = \eta^\alpha u_\alpha = r^2 \sin^2\theta \frac{d\phi}{d\tau} \qquad \rightarrow \qquad \ell = r^2 \frac{d\phi}{d\tau}$$

where we have exploited the spherical symmetry of Schwarzschild by adopting the value $\theta = \pi/2$

• What are these conserved quantities? ϵ is the energy per unit rest mass, and ℓ is the angular momentum per unit rest mass (which we can see from dimensional analysis, at low speeds and large distances).

• This is exactly the kind of useful thing we need to consider particle orbits — the conserved energy and angular momentum are one of the fundamental descriptors for orbits from classical mechanics.

• For our geodesic observer, there is a normalization constraint on the 4-velocity, $u^\alpha u_\alpha = -1$

• If we are confining our attention to the plane $\theta = \pi/2$, then the components of our 4-velocity are

$$u^\alpha = \left(u^t, u^r, u^\theta, u^\phi\right) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$$

• Using these components, we can write out the 4-velocity normalization in Schwarzschild as

$$-1 = u^\alpha u_\alpha = u^\alpha u^\beta g_{\alpha\beta} = -\left(1 - \frac{2M}{r}\right)(u^t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(u^r)^2 + r^2(u^\phi)^2$$

• Using our identities for e and ℓ, together with a bit of algebraic massage yields

$$\frac{\epsilon^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) - 1\right]$$

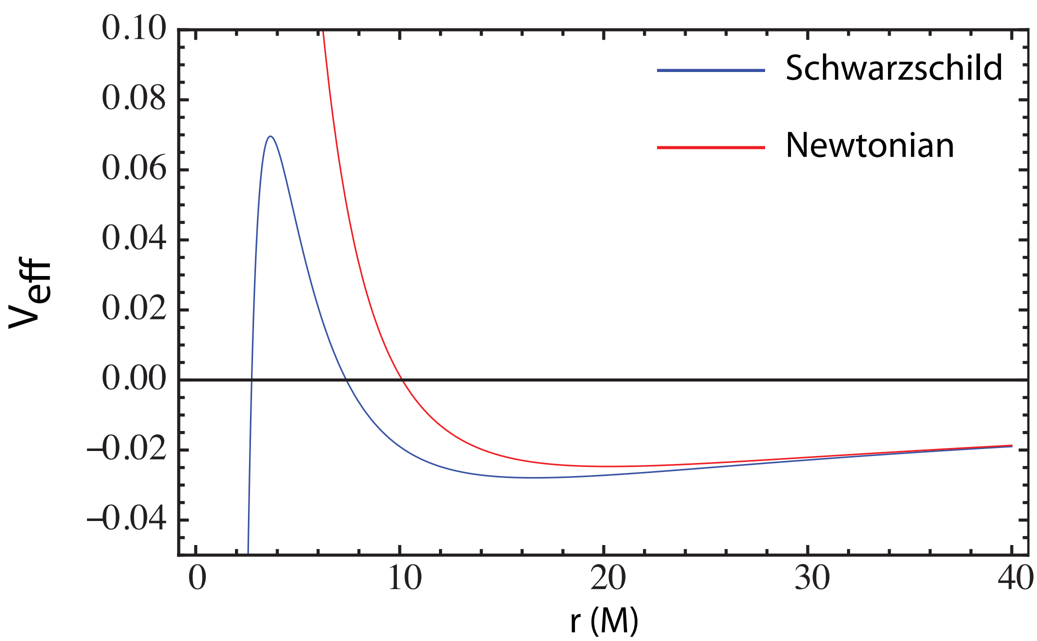
which it is conventional to write as

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

• The quantity $V_{eff}(r)$ is the *effective radial potential*; it is a very useful quantity to consider here because it is completely analogous to the effective potential from orbital theory in classical mechanics. If we fully expand $V_{eff}(r)$ we get

$$\begin{aligned} V_{eff}(r) &= \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) - 1\right] \\ &= \underbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}}_{Newtonian} - \underbrace{\frac{M\ell^2}{r^3}}_{correction} \end{aligned}$$

• The looks just like the classic effective potential from classical mechanics with a central angular momentum barrier, with an extra correction from general relativity. What does that correction do? Look at a graph (for $\ell/M = 4.5$, below)!



• So what do we notice right away?

▷ There is a minimum in the effective potential, just as in the Newtonian case. For a particle bound in this potential, there is a *stable circular orbit* at the minimum of the potential.

▷ There is a *maximum* in the effective potential, unlike in the Newtonian case. There is an *unstable circular orbit* at the maximum in the potential

▷ The *shape* of this potential is controlled by the size of the angular momentum, ℓ.

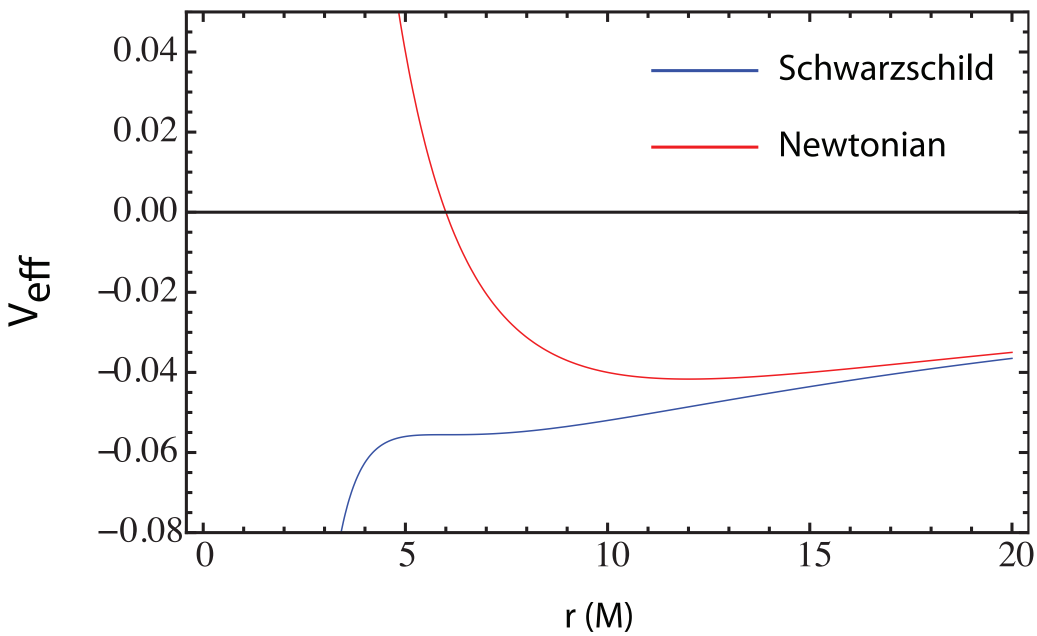
• We can find the min/max of the potential by constructing a radial derivative:

$$\frac{dV_{eff}}{dr} = 0 \qquad \rightarrow \qquad r_{\substack{min \\ max}} = \frac{\ell^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right]$$

• The absolute minimum value for this occurs when $\ell = \sqrt{12}M$, then

$$r_{min} = r_{ISCO} = 6M$$

This is known as the *ISCO* — the *innermost stable circular orbit*. The effective potential in this case looks like:



• *There are no stable circular orbits at radii smaller than* r_{isco}. This has important astrophysical consequences when considering phenomena such as accretion — material slowly works its way down the gravitational potential, giving up energy and angular momentum until it accretes onto the central, compact object. If there is a minimum radius at which material (gas, in the accretion case) can stably orbit a black hole, it will plunge at all smaller radii. This bounds the amount of gravitational binding energy that can be extracted by a particle.

Motion in the Effective Potential

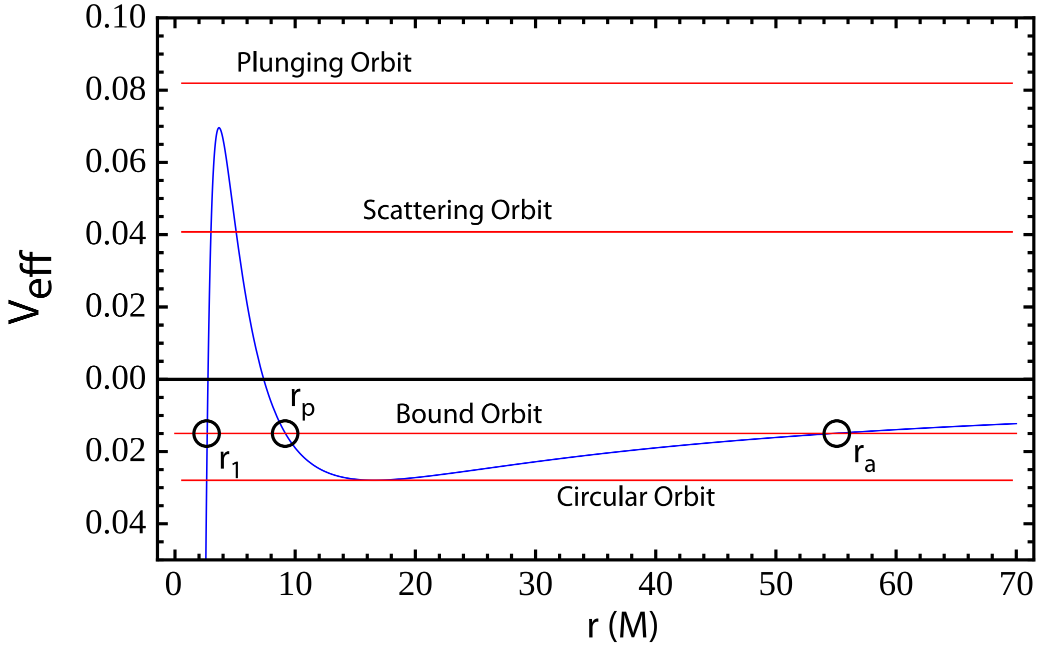
• The radial motion can be interpreted completely from the effective potential equation.

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \rightarrow \qquad \frac{dr}{d\tau} = \pm\sqrt{2}\,(\mathcal{E} - V_{eff})^{1/2}$$

• The angular evolution in time similarly can be found:

$$\ell = r^2\frac{d\phi}{d\tau} \qquad \rightarrow \qquad \frac{d\phi}{d\tau} = \frac{\ell}{r^2}$$

• From the energy viewpoint, and particle with energy constant $\mathcal{E}$ divides its energy between its potential energy and its kinetic energy. If you plot the particle's energy on the plot of $V_{eff}(r)$, then the gap between the line representing the energy and the line representing the effective potential is a measure of the kinetic energy of the particle.



• Where the energy line *crosses* the V_{eff} line, all of the kinetic energy has been dumped into potential energy, and the particle comes to rest. This is called a ***turning point***. The turning points can be found by solving the conditional equation

$$0 = V_{eff}(r) - \mathcal{E}$$

Caveat Physitor! This is a cubic equation in r, so there is, principle, closed form solutions for the turning points. They are however quite long and ugly to write out; it usually more expedient and simple to find the turning points numerically.

• There will be three or fewer turning points; the ones of interest are always real and $r > 0$. They are:

▷ The outermost turning point is called the *apoapsis* or *apocenter*, r_a. Outbound particles that reach the apocenter turn around, and move back toward the center of the potential.

▷ The turning point on the outside edge of the angular momentum barrier is called the *pericenter* or *periapsis*, r_p.

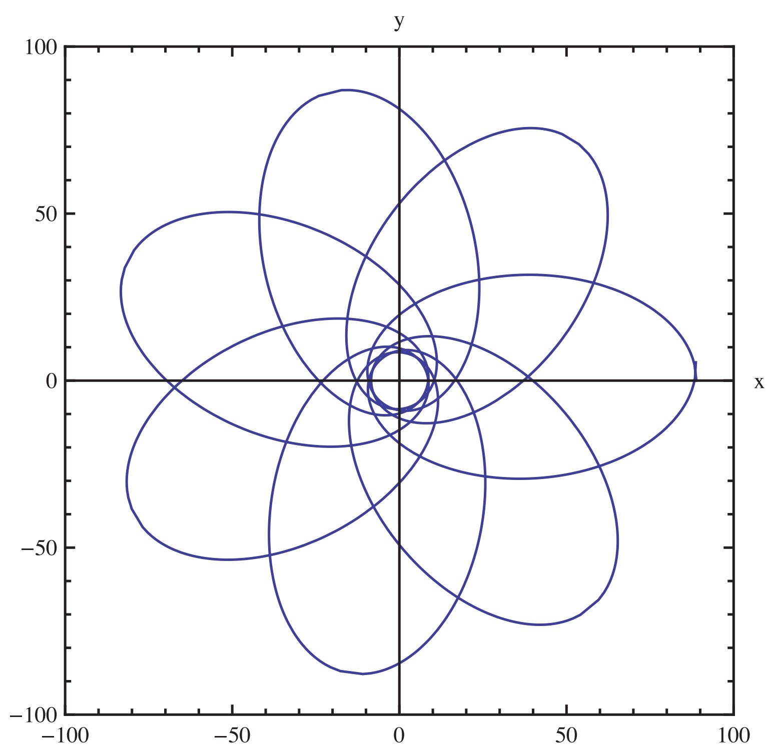
▷ The turning point on the inside edge of the angular momentum barrier is labeled r_1 or r_{in}. It is not a turning point of practical concern unless the body generating the potential is exceedingly compact (usually a black hole), as the particle would be *inside* to object otherwise! As a turning point of physical interest, it corresponds to a body deep in the gravitational well, rising away from the center to the turning point, then turning around and plunging back to center. There is no well established physical phenomenon with this behaviour.

• If we'd like to know the *shape* of the orbit, we need to know how ϕ varies with radial position r. We can work that out from $dr/d\tau$ and $d\phi/d\tau$:

$$\frac{d\phi}{dr} = \frac{d\phi/d\tau}{dr/d\tau} = \pm\frac{\ell}{\sqrt{2}r^2}\left(\mathcal{E} - V_{eff}\right)^{-1/2}$$

• Integrating this, ϕ is an ever increasing function that represents the total angle that has been swept out in the orbit.

• For bound orbits, numerical integration will make orbits like the one shown below



• Why is the orbit *precessing*? Because the particle is moving through the potential between r_p and r_a, it must be on an eccentric orbit. That means part of the orbit is spent deep down in the gravitational well, where space and time are stretched by the central mass M, and part of the time out near the rim of the well where space and time are stretched less. This stretching of the spacetime means that the particle accumulates ϕ at a varying rate, and accumulates $\phi = 2\pi$ well before it returns to its starting radius!

• This is the famous *perihelion precession* effect that Einstein suggested as a test of general relativity and was validated against the orbit of Mercury.

• We will forego the derivation of the perihelion precession rate and simply state it by fiat (you can find a derivation in Hartle's excellent book *Gravity* on pages 201-204, or in Chapter 11 of Schutz). The perihelion shift, in radians per orbit is:

$$\delta\phi = \frac{6\pi M}{a(1-e^2)}$$

where a is the semi-major axis, and e is the eccentricity of the orbit. As this value is often small, it is often expressed in units of "angle/century." If the system has an orbital period P and it is observed for some time T_{obs}, then the *observed periastron shift* will be

$$\delta\phi_{obs} = \frac{6\pi M}{a(1-e^2)}\frac{T_{obs}}{P}$$

• Now there is a curious exercise to be played here with the formula for $\delta\phi$ — let's compute the relativistic size of the perihelion shift for various astrophysical systems

System	***Central Mass***	$\boldsymbol{a}$	e	P_{orb}	$\delta\phi$
Mercury-Sun	$1M_\odot$	5.70×10^{10} m	0.2056	87.969 d	$0.4299''$/yr
PSR J0737-3039	$1.35 + 1.24M_\odot$	8.66×10^8 m	0.09	2.4 h	17.6°/yr
Earth-Moon	$1M_\oplus$	3.84×10^8 m	0.0549	27.32 d	$0.00027''$/yr
Jupiter-Io	1.899×10^{27} kg	4.22×10^8 m	0.0041	1.769 d	$2.68''$/yr
Jupiter-Europa	1.899×10^{27} kg	6.71×10^8 m	0.094	3.551 d	$0.84''$/yr
Jupiter-Amalthea	1.899×10^{27} kg	1.81×10^8 m	0.0032	43,043 s	$22.15''$/yr

Stellar Interiors

• In the vacuum outside a spherically symmetric star, the spacetime geometry is that of Schwarzschild. Imagine drawing a Gaussian sphere around the star. In this case, the Schwarzschild mass M is simply the mass of the star, enclosed inside the sphere.

• If we want to generalize to the *interior* of the star, one can argue that the correct modification to the metric components is to let $M \rightarrow m(r)$, where $m(r)$ is the mass enclosed inside the sphere or radius r.

• For reasons of convenience[1], it is useful to start with the metric in the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2$$

• The game will be to write and solve the Einstein Field Equations, to understand the dynamics of the mass the comprises the star under the influence of its own gravity. To do this, we need several pieces: the Einstein Tensor, $G_{\alpha\beta}$, the stress-energy tensor $T_{\alpha\beta}$. Additionally, we will insure that stress-energy is conserved, $T^{\alpha\beta}_{;\beta} = 0$, and specify an equation of state $p(\rho)$, that relates the pressure to the stellar density.

• In the form that we have written the metric above, the functions $\Phi(r)$ and $\Lambda(r)$ are simply two unknown functions. Given the high degree of symmetry in the spacetime, the Einstein Tensor only has 4 non-vanishing components. Using the shorthand $'$ to denote derivatives with respect to r, the surviving components of the Einstein Tensor are

$$G_{tt} = \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}\left[r\left(1 - e^{-2\Lambda}\right)\right]$$

$$G_{rr} = -\frac{1}{r^2}e^{2\Lambda}\left(1 - e^{-2\Lambda}\right) + \frac{2}{r}\Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}\left[\Phi'' + (\Phi')^2 + \frac{\Phi'}{r} - \Phi'\Lambda' - \frac{\Lambda'}{r}\right]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

• The simplest form of matter to consider is called a ***perfect fluid***. By definition, perfect fluids have the following properties:

▷ There is no heat conduction. This restriction amounts to the physical condition where energy flow is dictated by the flow of the fluid elements themselves. In the context of a stress-energy tensor, this means that all the time-space elements, $T^{0i} = T^{i0} = 0$.

▷ There is no viscosity. Viscosity is a manifestation of forces *parallel* to particle interfaces. This means that only forces perpendicular to particle interfaces are relevant, or that all $T^{ij} = 0$ for $i \neq j$.

• If a fluid element has 4-velocity u^α then the perfect fluid stress energy tensor is given by

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}$$

• Since the solutions of interest to us here are *static*, it must be that the 4-velocity of a fluid element has form

$$u^\alpha = \{u^t, 0, 0, 0\}$$

[1] In particular, with this definition of g^{tt}, the function Φ will be identified with the Newtonian gravitational potential.

• As you may be gathering from our repeated use of it, one of the most valuable tools in your arsenal is the normalization of the 4-velocity! It can be used to find values for unknown components of u^α. If we impose normalization on this 4-velocity we find

$$-1 = u^\alpha u^\beta g_{\alpha\beta} = (u^t)^2 g_{tt}$$

or

$$u^t = e^{-\Phi} \qquad\qquad u_t = -e^{\Phi}$$

• Using this, I can write out the components of the stress-energy tensor for a perfect fluid:

$$T_{tt} = (p+\rho)u_t u_t + p g_{tt} = \rho e^{2\Phi}$$

$$T_{rr} = (p+\rho)u_r u_r + p g_{rr} = p e^{2\Lambda}$$

$$T_{\theta\theta} = (p+\rho)u_\theta u_\theta + p g_{\theta\theta} = p r^2$$

$$T_{\phi\phi} = (p+\rho)u_\phi u_\phi + p g_{\phi\phi} = p r^2 \sin^2\theta = \sin^2\theta T_{\theta\theta}$$

• We have all the pieces we need to write out the EFEs. It is convenient (for physical interpretation) to adopt the unknown function $m(r)$ in lieu of the unknown function $\Lambda(r)$:

$$m(r) = \frac{1}{2} r \left(1 - e^{-2\Lambda}\right) \qquad \rightarrow \qquad e^{2\Lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

• The EFEs are $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. Using the identities above we get for G_{tt}

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho$$

and for G_{rr}

$$\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r[r - 2m(r)]}$$

• We have 4 unknowns: $\Phi(r), m(r), p(r)$, and $\rho(r)$. That means we need at least 4 equations to specify all the unknowns. We could use the $G_{\theta\theta}$ EFE, but have you seen it? I wouldn't want to mess with it unless I had to. So it behooves me to look for something simpler to work with.

• We *always* have the conservation of stress energy! So:

$$T^{\alpha\beta}_{\ ;\beta} = 0$$

Because of the symmetry of the spacetime (static and spherically symmetric), the only non-zero derivative must be $\beta = r$, so

$$T^{rr}_{\ ;r} = 0 \qquad \rightarrow \qquad (\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}$$

This is saying that the gravitational gradient (LHS) requires a pressure gradient (RHS) to balance out the influence of gravity and maintain the symmetries of the spacetime.

• For the last of our needed equations, we impose an *Equation of State*, a relationship between p and ρ:

$$p = p(\rho)$$

• Two of our four equations depend on $d\Phi/dr$; if we eliminate $d\Phi/dr$ between them we obtain the *Tolman-Oppenheimer-Volkoff Equation (TOV)*:

$$\frac{dp}{dr} = -\frac{(p+\rho)(m+4\pi r^3 p)}{r(r-2m)}$$

• What is the TOV Equation? I like to approach this question by taking the Newtonian limit: $m \ll r$ and $p \ll \rho$, then it becomes

$$\frac{dp}{d\rho} = -\frac{\rho m}{r^2}$$

which is just the usual equation of hydrostatic equilibrium from classical fluid dynamics!

Equation of State

• The equation of state (EOS) is the last fundamental expression that must be provided. It is an expression that describes how the pressure varies with energy density, $p = p(\rho)$. We assume such a relationship always exists, but that it can be different for different fluids that might comprise our relativistic star, so there is some breadth of choice here

• The simplest choice is for an ***incompressible star***, where the density is constant throughout: $\rho = const$. While we do this here for a relativistic star using the relativistic structure equations, most of you will be familiar with this choice as it is the usual choice made in introductory physics when studying fluid dynamics with ordinary fluids like water.

• Another popular choice for astrophysicists is a *polytropic equation of state*.

• A *polytrope* is a sphere of gas with the pressure P related to the density ρ by[2]

$$P = \kappa\rho^\gamma \qquad \rightarrow \qquad \gamma \equiv \frac{n+1}{n} \qquad \rightarrow \qquad P = \kappa\rho^{(n+1)/n}$$

where κ is a constant throughout the star (but could be different for different stars), and n is called the *polytropic index*. Polytropes are very useful in all kinds of stellar modeling, depending on the value of the index γ.

• What is this κ constant? Remember that κ is constant throughout the star, so imagine looking at the equation of state at the core where $P = P_c$ and $\rho = \rho_c$. Now look at P and ρ

[2] Remember that $P = P(r)$ and $\rho = \rho(r)$; I have suppressed the (r) here for clarity.

at any other radius r in the star. I can write the ratio of the equation of states as (note the κ cancel out)

$$\frac{P(r)}{P_c} = \left(\frac{\rho(r)}{\rho_c}\right)^{(n+1)/n} \qquad \rightarrow \qquad P(r) = \left(\frac{P_c}{\rho_c^{(n+1)/n}}\right)\rho(r)^{(n+1)/n}$$

• Comparing this to the polytrope equation, we note that physically κ characterizes the central pressure and central density in our model. Various "good" choices of polytropic index κ exist for different stars:

▷ $\kappa = 3$, "ordinary" main sequence stars, like the Sun (usually well described by ordinary fluid dynamics, without the need for relativistic treatments), as well as the degenerate *cores* of relativistic stars like white dwarfs
▷ $\kappa = 1.5$, Convective stars like red giants, or Jovian type planets
▷ $0.5 \lesssim \kappa \lesssim 1$, neutron stars. There is no definitive value that is good for neutron stars because there is not a known equation of state that always applies to these objects.

• Polytropes can also be used to describe other "gaseous" systems, like globular star clusters.

Integrating the Structure Equations

• Any time you integrate a system of differential equations, you must make physical considerations that give you an initial starting point for the endeavour (*initial data* or *boundary conditions*).

• We have two first order differential equations, for dp/dr and for dm/dr, so we chose *physical conditions* for these variables. In particular,

$$m(r=0) = 0 \qquad\qquad p(r=0) = p_c$$

The mass goes to zero at $r = 0$, and the pressure goes to some finite central value p_c (or alternatively, since we will specify an equation of state, there is a central density ρ_c).

• We have additional boundary conditions at the surface of the star. Here, $p(R) = 0$ (when I'm outside the star, the pressure goes to zero!).

• If we also insist that the metric components are continuous across the surface of the star, then looking at g_{rr} means inside the star we have

$$g_{rr}^{inside} = \left(1 - \frac{2m(r)}{r}\right)^{-1}$$

and outside the star we have

$$g_{rr}^{outside} = \left(1 - \frac{2M}{r}\right)^{-1}$$

Since these two metric components must be *equal* at $r = R$ we deduce

$$M = m(R)$$

• At this point you also have the machinery to address the question about the relationship between the stellar radius R and the Schwarzschild radius, $r_s = 2m(R) = 2M$. Can a fluid star, described by the Schwarzschild spacetime, have $R = 2M$?

• To address this, consider a star which has a radius near the Schwarzschild radius, so that $\epsilon = r - 2m(r)$. Here $\epsilon \ll 1$ and decreases with r.

• With this identity, the TOV equation goes like $1/\epsilon$. That means the *pressure gradient is negative* and is *huge* because ϵ is small. That means in this regime

- ▷ Any finite value of the pressure is rapidly driven to zero.
- ▷ The pressure is finite, so $p \to 0$ faster than $\epsilon \to 0$
- ▷ By definition, $p = 0$ at $r = R$, the surface of the star. This occurs *before* $R = 2m(r)$.