Midterm Exam

February 27, 2013

Take home exam. Use only the formulas provided here. You may contact me with any questions at *jim.wheeler@usu.edu* I will give you whatever help you need to move forward with your solution. Use no other sources, and carry out the calculation by hand.

Problem 1: Compute the Schwarzschild metric by solving the Einstein equation.

You may assume the following static, spherically symmetric form for the line element ($c = 1$):

$$ds^2 = -f^2(r)\,dt^2 + g^2(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

This line element describes a whole class of Riemannian geometries, all of which are static and spherically symmetric. Your problem is to compute the Riemann curvature tensor for all metrics in this class, in terms of derivatives of the functions f and g, then impose the Einstein equation on those curvatures and solve for the (nearly) unique functions f and g that satisfy it.

Follow the following steps:

1. From the line element, write the metric tensor, $g_{\alpha\beta}$, and its inverse, $g^{\alpha\beta}$.
2. From the metric and its inverse, compute the Christoffel connection, using the formula
$$\Gamma^{\alpha}_{\ \mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right)$$
Carefully check and tabulate all nonzero components.
3. Compute the Riemann curvature tensor, given by
$$R^{\alpha}_{\ \beta\mu\nu} = \Gamma^{\alpha}_{\ \beta\mu,\nu} - \Gamma^{\alpha}_{\ \beta\nu,\mu} + \Gamma^{\alpha}_{\ \rho\nu}\Gamma^{\rho}_{\ \beta\mu} - \Gamma^{\alpha}_{\ \rho\mu}\Gamma^{\rho}_{\ \beta\nu}$$
4. Compute the Ricci tensor, given by
$$R_{\beta\mu} = R^{\alpha}_{\ \beta\alpha\nu}$$
5. The Einstein equation is
$$G^{\alpha\beta} = \kappa T^{\alpha\beta}$$
In vacuum, for example, outside a static, spherically symmetric star, $T^{\alpha\beta} = 0$, so we have
$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 0$$
where $R = g^{\alpha\beta} R_{\alpha\beta}$ is the Ricci scalar. If we contract $G^{\alpha\beta}$ with the metric, we have
$$\begin{aligned} 0 &= g_{\alpha\beta} G^{\alpha\beta} \\ &= g_{\alpha\beta} R^{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\alpha\beta} R \\ &= R - \frac{1}{2}\cdot 4R \\ &= -R \end{aligned}$$

so that for vacuum solutions, the Ricci scalar vanishes. Then the Einstein equation becomes

$$R_{\alpha\beta} = 0$$

From your answer for part 4, write these equations. There will be a maximum of 10, second order, differential equations, but most vanish identically and you will have only a few second order equations for f and g. Solve them.

6. Write the final line element, putting your solutions for $f(r)$ and $g(r)$ into

$$ds^2 = -f^2(r)\,dt^2 + g^2(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

This is the Schwarzschild line element. It describes the curved spacetime outside a static, spherical mass.