Manifolds, vectors and forms

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1 Manifolds

Loosely speaking, a manifold is a (topological) space that looks like a small piece of R^n in any sufficiently small region. For example, the 2-dimensional surface of a ball in 3-dimensions it the space S^2. If we move very close to the surface, it looks like a piece of a Euclidean plane. Indeed, the distance between two nearby points on the surface of a sphere of radius R is

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Pick a point, $x^i = (\theta_0, \varphi_0)$, and look in a nearby region of the surface. Expanding about that point, in a region $(\theta_0 + \varepsilon, \varphi_0 + \sigma)$,

$$\begin{aligned} \sin(\theta_0 + \varepsilon) &= \sin\theta_0 + \cos\theta_0 \varepsilon + \dots \\ d\theta &= d\varepsilon \\ d\varphi &= d\sigma \end{aligned}$$

the distance becomes

$$\begin{aligned} ds^2 &= R^2 \left(d\varepsilon^2 + (\sin\theta_0 + \cos\theta_0\varepsilon + \dots)^2 d\sigma^2\right) \\ &\approx R^2 \left(d\varepsilon^2 + \sin^2\theta_0 d\sigma^2\right) \end{aligned}$$

Now define new coordinates

$$\begin{aligned} x &= R\varepsilon \\ y &= R\sin\theta_0 d\sigma \end{aligned}$$

and as long as we can ignore the terms of order $\varepsilon (d\sigma)^2$, we have

$$ds^2 \approx dx^2 + dy^2$$

The sphere looks like a plane when we get close enough.

In general, we describe this sort of procedure in terms of mappings. In order for a space to be an n-dimensional, C^∞ manifold, $\mathcal{M}^n$, we require that in a neighborhood of each point, $N(\mathcal{P})$, there must be a $1-1$, onto mapping, ϕ, to an open subset, $O(\mathbf{x}_0)$, in R^n, where we take $\phi(\mathcal{P}) = \mathbf{x}_0$

$$\phi : N(\mathcal{P}) \leftrightarrow O(\mathbf{x}_0)$$

For any point in the open set $\mathcal{Q} \in N(\mathcal{P})$, there is a unique $\mathbf{x} \in O(\mathbf{x}_0)$ with

$$\phi(\mathcal{Q}) = \mathbf{x}$$

In R^n, $\mathbf{x}$ is an n-tuple of numbers and these are the coordinates of the point $\mathcal{Q}$. The mapping ϕ is called a *chart*.

We cannot usually find a single such a mapping ϕ that assigns coordinates to every point $\mathcal{M}^n$, so we need to specify how the coordinates in nearby regions are related. Let $\mathcal{P}_1, \mathcal{P}_2$ be points in $\mathcal{M}^n$ with neighborhoods $N_1(\mathcal{P}_1)$ and $N_2(\mathcal{P}_2)$ and charts ϕ_1 and ϕ_2. Then for all points in the intersection

$$N_{12} = N_1(\mathcal{P}_1) \cap N_2(\mathcal{P}_2)$$

we have two different sets of coordinates, $O_1 = \phi_1(N_{12})$ and $O_2 = \phi_2(N_{12})$. We want there to exist a sensible transformation between these.

Consider the case of polar coordinates and Cartesian coordinates for the plane. These are related by

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ r &= \sqrt{x^2+y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

Except at the origin, these relations are differentiable functions. In fact, they are infinitely differentiable (smooth). This is the condition we will require, but we need to figure out how to require it. The relationship must hold between the coordinates $\phi_1(N_{12})$ and $\phi_2(N_{12})$, and we can specify the relationship by using the inverse mapping, ϕ_1^{-1}. We apply two maps in succession:

$$\begin{aligned} \phi_1^{-1}: \ O_1 &\rightarrow N_{12} \\ \phi_2: \ N_{12} &\rightarrow O_2 \end{aligned}$$

The combination of these, $\phi_2 \circ \phi_1^{-1}$ is a map between two open sets in R^n,

$$\phi_2 \circ \phi_1^{-1}: \ O_1 \rightarrow O_2$$

We require the mapping $\phi_2 \circ \phi_1^{-1}$ from $O_1 \subset R^n$ to $O_2 \subset R^n$ to be infinitely differentiable. $\mathcal{M}^n$ is then a C^∞ manifold.

2 Vectors and forms

We can now define two vector spaces associated with any manifold. Both spaces depend on two simple ideas: functions and curves.

A real-valued function on a manifold is an assignment of a real number to each point of the manifold,

$$f: \mathcal{M} \rightarrow R$$

By using the charts of the manifold, we can differentiate the function. For any point $\mathcal{P}$ of $\mathcal{M}$, there exists a chart on a neighborhood, $N(\mathcal{P})$, of $\mathcal{P}$,

$$\phi: N(\mathcal{P}) \leftrightarrow O(\mathbf{x}_0)$$

so combining with the function for each point in $N(\mathcal{P})$ we have a mapping from a region in R^n to the reals,

$$f \circ \phi^{-1}: O(\mathbf{x}_0) \rightarrow R$$

We may write the result of this as map as the number $f(\mathbf{x}) \in R$, where $\mathbf{x}$ is a point in R^n. Then f is a real-valued function on R^n and we may differentiate it in the usual way,

$$\frac{\partial f}{\partial x^\alpha}$$

While functions map from $\mathcal{M}$ to R, a curve is a mapping from R into $\mathcal{M}$:

$$C: R \rightarrow \mathcal{M}$$

Combined with a chart

$$\phi^{-1} \circ C: \ R \rightarrow O(\mathbf{x}_0)$$

we have a parameterized curve in R^n, $\mathbf{x}(\lambda)$, where as $\lambda \in R$ varies, the point $\mathbf{x}(\lambda)$ traces out a path in R^n.

2.1 Vectors

We define:

Def: A vector at a point $\mathcal{P}$ is a directional derivative at $\mathcal{P}$ Consider the values of a function $f(\mathcal{P})$ restricted to a curve $C(\lambda)$, $f(C(\lambda))$. The derivative

$$\frac{df}{d\lambda}$$

is intrinsic to the space. The function at any point of the curve C is a number, and our usual definition of derivative works:

$$\frac{df}{d\lambda} = \lim_{\varepsilon\to 0}\frac{f(C(\lambda+\varepsilon)) - f(C(\lambda))}{\varepsilon}$$

Here, λ and ε are real numbers, $C(\lambda)$ and $C(\lambda+\varepsilon)$ are points of the manifold, and $f(C(\lambda))$ is another number, the value of the function f at the point $C(\lambda)$.

If we use a chart, we may write

$$\begin{aligned} f(\mathcal{P}) &= f\circ\phi^{-1}\circ\phi(\mathcal{P}) \\ &= \left(f\circ\phi^{-1}\right)(\phi(\mathcal{P})) \\ &= F(x^\alpha) \end{aligned}$$

Here, $f\circ\phi^{-1}$ maps a point with coordinates x^α in R^n to a point $\mathcal{P}$ of the manifold, then f evaluates on that point. Then, to evaluate the derivative along C, where

$$\phi(C(\lambda)) = x^\alpha(\lambda)$$

we have

$$\begin{aligned} \frac{df}{d\lambda} &= \lim_{\varepsilon\to 0}\frac{f(C(\lambda+\varepsilon)) - f(C(\lambda))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f\circ\phi^{-1}\circ\phi(C(\lambda+\varepsilon)) - f\circ\phi^{-1}\circ\phi(C(\lambda))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f(x^\alpha(\lambda+\varepsilon)) - f(x^\alpha(\lambda))}{\varepsilon} \\ &= \frac{df(x^\alpha(\lambda))}{d\lambda} \\ &= \frac{\partial f(x^\alpha)}{\partial x^\alpha}\frac{dx^\alpha(\lambda)}{d\lambda} \end{aligned}$$

Therefore, in coordinates, i.e., a basis, we may write

$$\frac{d}{d\lambda}f(\lambda) = \frac{dx^\alpha}{d\lambda}\frac{\partial}{\partial x^\alpha}f(x^\alpha)$$

We now define a *vector* to be the directional derivative operator

$$\frac{d}{d\lambda} = \frac{dx^\alpha}{d\lambda}\frac{\partial}{\partial x^\alpha}$$

We see that once we make a choice of coordinates x^α, all directional derivatives may be written as linear combinations of the *basis vectors*

$$\vec{e}_\alpha = \frac{\partial}{\partial x^\alpha}$$

The coefficients of this linear combination are the tangents to the mapped curve, $x^{\alpha}(\lambda) = \phi \circ C(\lambda)$,

$$v^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$$

We now show that these directional derivatives form a vector space. It is not hard to see that scalar multiples are also curves, since a change of parameter from λ to $a\lambda$ changes v^{α} to av^{α} so that av^{α} is also a vector. The only tricky part of the demonstration is to show that we can add directional derivatives to get a third directional derivative. We content ourselves with demonstrating this.

Suppose we have two curves, $C_1(\lambda)$ and $C_2(\lambda)$. Then

$$\begin{aligned} \phi \circ C_1 &= x_1^{\alpha}(\lambda) \\ \phi \circ C_2 &= x_2^{\alpha}(\lambda) \end{aligned}$$

where $x_1^{\alpha}(\lambda), x_2^{\alpha}(\lambda)$ are two curves in R^n. Since R^n is a vector space, we may add the vectors x_1^{α} and x_2^{α} at each value of λ to get a new curve

$$x_3^{\alpha}(\lambda) = (x_1^{\alpha} + x_2^{\alpha})(\lambda)$$

Then

$$\begin{aligned} C_3(\lambda) &= \phi^{-1}(x_3^{\alpha}(\lambda)) \\ &= \phi^{-1}(\phi \circ C_1 + \phi \circ C_2) \end{aligned}$$

is a curve in $\mathcal{M}$. Since $\phi \circ C_3$ is just $x_3^{\alpha}(\lambda)$, the directional derivative along C_3 is

$$\begin{aligned} \frac{d}{d\lambda}_{(3)} &= \frac{dx_3^{\alpha}(\lambda)}{d\lambda}\frac{\partial}{\partial x^{\alpha}} \\ &= \frac{d(x_1^{\alpha} + x_2^{\alpha})(\lambda)}{d\lambda}\frac{\partial}{\partial x^{\alpha}} \\ &= \frac{dx_1^{\alpha}}{d\lambda}\frac{\partial}{\partial x^{\alpha}} + \frac{dx_2^{\alpha}}{d\lambda}\frac{\partial}{\partial x^{\alpha}} \\ &= \frac{d}{d\lambda}_{(1)} + \frac{d}{d\lambda}_{(2)} \end{aligned}$$

The sum of two directional derivatives is therefore a third directional derivative. This, together with the usual properties of addition and scalar multiplication, show that directional derivatives form a vector space.

2.2 Forms

There is a second vector space arising from curves and functions on a manifold.

Def: A form is a linear map on curves The basic idea here is that an integral is a linear mapping. If we integrate the differential of a function along a curve, we get a number,

$$f(x^{\alpha}) = \int_C^{x^{\alpha}} df$$

The differentials, df, combine linearly,

$$a\,f(x^{\alpha}) + b\,g(x^{\alpha}) = \int_C^{x^{\alpha}} (a\,df + b\,dg)$$

that is, if we regard df, dg as mappings that take the curve C into the reals, R, then $a\,df + b\,dg$ is another such mapping. What we need to do is define these things in a way that applies to general manifolds.

Consider an arbitrary linear mapping on curves,

$$\tilde{\omega} : C \to R$$

Linearity guarantees that for any such mapping, we can divide the curve $C(\lambda)$ into small pieces,

$$C_k(\lambda) = \{C(\lambda) \mid \lambda \in [\lambda_k, \lambda_{k+1}]\}$$

so that C_k is the piece of C running from parameter values λ_k to λ_{k+1}. Clearly,

$$C(\lambda) = \sum_{k=0}^{n} C_k(\lambda)$$

and by the linearity of $\tilde{\omega}$,

$$\tilde{\omega}(C(\lambda)) = \sum_{k=0}^{n} \tilde{\omega}(C_k(\lambda))$$

Now use charts to write this in coordinates:

$$\tilde{\omega}(C(\lambda)) = \sum_{k=0}^{n} \tilde{\omega} \circ \phi^{-1} \circ \phi \circ C_k(\lambda)$$

where

$$\begin{aligned} \tilde{\omega} \circ \phi^{-1} : R^n &\to R \\ \phi \circ C_k(\lambda) : R &\to R^n \end{aligned}$$

so that the right side is a mapping from points along a curve, $x^\alpha(\lambda)$, in R^n to the reals, R, giving a function, $f(\lambda)$.

Now consider the sum,

$$\sum_{k=0}^{n} \tilde{\omega} \circ \phi^{-1} \circ \phi \circ C_k(\lambda)$$

Let n become large so that $\lambda_{k+1} - \lambda_k \to d\lambda$. Then $\phi \circ C_k(\lambda)$ is just the coordinate change, $dx^\alpha(\lambda) = \frac{dx^\alpha}{d\lambda} d\lambda$, for an infinitesimal piece of the curve from λ to $\lambda + d\lambda$. The form returns the value a real number which must depend linearly on this coordinate displacement,

$$\tilde{\omega} \circ \phi^{-1}(dx^\alpha) = \omega_\alpha dx^\alpha$$

In the limit, the sum becomes an integral along the curve C,

$$\tilde{\omega}(C(\lambda)) = \int_C \omega_\alpha dx^\alpha$$

Again, we make an operator interpretation. The form is the integrand,

$$\tilde{\omega} = \omega_\alpha dx^\alpha$$

The components of the form are ω_α and the coordinate differentials dx^α form a basis. The operation of $\tilde{\omega}$ on any curve C is the integral of $\tilde{\omega}$ along the curve. We may also write the integral as an integral over λ,

$$\begin{aligned} \tilde{\omega}(C(\lambda)) &= \int_C \omega_\alpha \frac{dx^\alpha}{d\lambda} d\lambda \\ &= \int_C f(\lambda)\, d\lambda \end{aligned}$$

where

$$f(\lambda) = \omega_\alpha \frac{dx^\alpha}{d\lambda}$$

is a mapping to the reals, given by combining the form with a tangent vector. This is just the mapping of $\tilde{\omega}$ on an infinitesmal curve $\frac{dx^\alpha}{d\lambda} d\lambda$. Writing the tangent vector in components, $\phi(\vec{t}) = \frac{dx^\alpha}{d\lambda}$, we may also write the form as a linear mapping on vectors,

$$\tilde{\omega}\left(\vec{t}\right) = \omega_\alpha \frac{dx^\alpha}{d\lambda}$$

This expression, like $\tilde{\omega}(C)$, is independent of the basis.