What is a tensor?

Here we give a quick-start introduction to the ideas surrounding the thing called a "tensor". We have seen that associated to every vector space V there is another vector space, the dual vector space V^*, defined as the set of linear functions on V. The evaluation of a dual vector $\underline{\alpha} \in V^*$ on a vector $\vec{v} \in V$ is denoted by $\underline{\alpha}(\vec{v})$. In terms of a basis $\vec{e}_i$ for V and a dual basis $\underline{\omega}^i$ for V^* we have

$$\underline{\omega}^i(\vec{e}_j) = \delta^i_j, \quad \vec{v} = v^i \vec{e}_i, \quad \underline{\alpha} = \alpha_i \underline{\omega}^i,$$

$$\underline{\alpha}(\vec{v}) = \alpha_i v^i.$$

From this last formula you can see that, for a given dual vector $\underline{\alpha}$, $\underline{\alpha}(\vec{v})$ is a linear function of $\vec{v}$. But, if you can see that, you can also see that $\underline{\alpha}(\vec{v})$ is, for a given vector $\vec{v}$, a linear function of $\underline{\alpha}$. Thus we can define V as the set of linear functions on V^*! Put differently, $V^{**} = V$.

We generalize this point of view to define tensors. A tensor T of type $\binom{p}{q}$ is a mutil-linear function assigning a scalar to q vectors and p dual vectors. The set of all such multi-linear functions defines a vector space. We now want to explore this in some more detail.

The tensor product

The idea behind the tensor product is that vectors and dual vectors, which define linear maps, can be juxtaposed to define multi-linear maps. For example, given a pair of dual vectors $\underline{\alpha}$ and $\underline{\beta}$ we can define a bilinear function on two copies of V via

$$\vec{v}, \vec{w} \longrightarrow \underline{\alpha}(\vec{v})\underline{\beta}(\vec{w}).$$

This bilinear function is denoted by $\underline{\alpha} \otimes \underline{\beta}$. We have in, a given basis,

$$\underline{\alpha} \otimes \underline{\beta}(\vec{v}, \vec{w}) = \underline{\alpha}(\vec{v})\underline{\beta}(\vec{w}) = \alpha_i \beta_j v^i w^j.$$

The funny $\otimes$ symbol is called the *tensor product*. It amounts to making an ordered pair of two quantities, dual vectors in this case. So, the tensor product of two dual vectors is just "two dual vectors at once". The tensor product of two vectors is just two vectors at once, which define a bilinear function on two dual vectors:

$$\vec{v} \otimes \vec{w}(\underline{\alpha}, \underline{\beta}) = \underline{\alpha} \otimes \underline{\beta}(\vec{v}, \vec{w}) = \alpha_i \beta_j v^i w^j.$$

Now you might be able to guess the meaning of something like

$$\underline{\alpha} \otimes \vec{v} = \alpha_i v^j \underline{\omega}^i \otimes \vec{e}_j.$$

We have

$$\underline{\alpha} \otimes \vec{v}(\vec{w}, \underline{\beta}) = \alpha_i w^i v^j \beta_j.$$

Evidently we can juxtapose as many vectors and dual vectors as we like. A tensor T of type $\binom{p}{q}$ is a multi-linear function of q vectors and p dual vectors, which makes a scalar out of them. The tensor can be constructed by juxtaposing (with the tensor product) p vectors and q dual vectors (and adding them, see below).

Scalar multiplication and addition of tensors

We can do two more things. First, we can define scalar multiplication of tensors. For example, take the tensor $\vec{v} \otimes \vec{w}$ and a scalar a, we *define*

$$a(\vec{v} \otimes \vec{w}) = (a\vec{v}) \otimes \vec{w} = \vec{v} \otimes (a\vec{w}).$$

Second we define addition of two tensors – of the same type – much as we did with dual vectors to be another multilinear map of the same type. For example,

$$(\underline{\alpha} \otimes \underline{\beta} + \underline{\gamma} \otimes \underline{\delta})(\vec{v}, \vec{w}) = \underline{\alpha}(\vec{v})\underline{\beta}(\vec{w}) + \underline{\gamma}(\vec{v})\underline{\delta}(\vec{w}) = \alpha_i v^i \beta_j v^j + \gamma_i v^i \delta_j w^j.$$

In this way the set of tensors of type $\binom{p}{q}$ becomes a vector space.

Being a vector space, these tensor product spaces have a basis. Happily, they are just the tensor products of the vector and dual vector bases. For example, it is not hard to check that *any* tensor T of type, say, $\binom{2}{0}$ can be written

$$T = T^{ij} \vec{e}_i \otimes \vec{e}_j.$$

So a such tensors are determined by a choice of basis and a square array of numbers. Similarly, tensors of type $\binom{0}{2}$ can be written in terms of a square array as:

$$H = H_{ij} \underline{\omega}^i \otimes \underline{\omega}^j.$$

Finally tensors of type $\binom{1}{1}$ can be written in terms of yet another square array.

$$K = K^i_j \underline{\omega}^j \vec{e}_i.$$

While all three of these tensors are determined by square arrays of numbers, they have very different meanings, as the index placement is supposed to help remind you. I think you can see how to generalize all this to tensors of any type.

Notice that the components of the tensorcT are just c times the components of T, for example,

$$(cT)_{ij} = cT_{ij}.$$

The components of $T + W$ are just the sum of the components of T and W, *e.g.,*

$$(T + W)_{ij} = T_{ij} + W_{ij}.$$

Tensor fields are defined much as we define (dual) vector fields. A tensor *field* of type $\binom{p}{q}$ assigns a tensor of type $\binom{p}{q}$ to each point of your underlying space or spacetime.

Contraction of tensors

We've defined the tensor product. And we've defined the sum and scalar multiples of tensors. A final operation must be introduced. It is called *contraction.* When ever a vector $\vec{v}$ is evaluated on a form $\underline{\alpha}$, or vice versa, we say that we have contracted the two quantities. In terms of indices, contraction amounts to setting two indices equal and summing. For example, the contraction of $\vec{v}$ and $\underline{\alpha}$ is given by

$$\vec{v}(\underline{\alpha}) = v^i \alpha_j.$$

Since all tensors can be formed by products of vectors and dual vectors, all contractions can be obtained by setting one or more pairs of indices equal – one up and one down – and performing the indicated sums according to the Einstein summation convention.

The dot product

All the proceeding was pretty abstract and might have made your head spin. Let's look at a very simple example. Consider 3-d space, viewed just as a vector space with basis:

$$\vec{e}_i = (\hat{\imath}, \hat{\jmath}, \hat{k}).$$

The dot product, as you know, is defined to take two vectors

$$\vec{v} = v^i \vec{e}_i, \quad \vec{w} = w^i \vec{e}_i,$$

and make a scalar:

$$\vec{v} \cdot \vec{w} = v^1 w^1 + v^2 w^2 + v^3 w^3.$$

Evidently, this is a bilinear function of the two vectors. There must be a corresponding tensor of type $\binom{0}{2}$. This tensor is called the metric. It is given by

$$g = \delta_{ij} \underline{\omega}^i \underline{\omega}^j.$$

We have

$$g(\vec{v}, \vec{w}) = \delta_{ij} v^i w^j = \vec{v} \cdot \vec{w}.$$

The components of the metric in this example constitute the identity matrix. In general this won't be true. What *is* true is that a definition of scalar product is equivalent to defining a tensor of type $\binom{0}{2}$ whose components form a symmetric square array.

It is a useful exercise to view the dot product from yet another sophisticated point of view. To do this we write

$$\delta_{ij} v^i w^j = (\delta_{ij} v^i) w^j = \alpha_j w^j,$$

where

$$\alpha_j = \delta_{ij} v^i.$$

Evidently, we can define a dual vector

$$\underline{\alpha} = \alpha_i \underline{\omega}^i,$$

and write

$$\vec{v} \cdot \vec{w} = \underline{\alpha}(\vec{w}).$$

The process of using the metric to define a dual vector from a vector is called "lowering the index" on the vector. You can see that this process involves a contraction between one of the dual vectors making up the metric tensor and the vector.