Index Notation

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One of the hurdles to learning general relativity is the use of vector indices as a calculational tool. While you will eventually learn tensor notation that bypasses some of the index usage, the essential form of calculations often remains the same.

Index notation allows us to do more complicated algebraic manipulations than the vector notation that works for simpler problems in Euclidean 3-space. Even there, many vector identities are most easily established using index notation. When we begin discussing 4-dimensional, curved spaces, our reliance on algebra for understanding what is going on is greatly increased. We cannot make progress without these tools.

1 Three dimensions

To begin, we translated some 3-dimensional formulas into index notation. You are familiar with writing boldface letters to stand for vectors. Each such vector may be expanded in a basis. For example, in our usual Cartesian basis, every vector $\mathbf{v}$ may be written as a linear combination

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

We need to make some changes here:

1. Replace the x, y, z labels with a numbers $1, 2, 3$:

$$(v_x, v_y, v_z) \longrightarrow (v_1, v_2, v_3)$$

2. Write the labels raised instead of lowered.

$$(v_1, v_2, v_3) \longrightarrow \left(v^1, v^2, v^3\right)$$

3. Use a lower-case Latin letter to run over the range, $i = 1, 2, 3$. This means that the single symbol, v^i stands for *all three* components, depending on the value of the index, i.

4. Replace the three unit vectors by an indexed symbol:

$$\hat{\mathbf{e}}_i$$

   so that $\hat{\mathbf{e}}_1 = \hat{\mathbf{i}}, \hat{\mathbf{e}}_2 = \hat{\mathbf{j}}$ and $\hat{\mathbf{e}}_3 = \hat{\mathbf{k}}$

With these changes the expansion of a vector $\mathbf{v}$ simplifies considerably because we may use a summation:

$$\begin{aligned} \mathbf{v} &= v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}} \\ &= v^1\hat{\mathbf{e}}_1 + v^2\hat{\mathbf{e}}_2 + v^3\hat{\mathbf{e}}_3 \\ &= \sum_{i=1}^{3} v^i\hat{\mathbf{e}}_i \end{aligned}$$

We have chosen the index positions, in part, so that inside the sum there is one index up and one down. We will continue this convention, noting only that there is a deeper reason for the distinction between the index positions, to be discussed later.

Linear combinations of vectors are also vectors. Thus, if we have two vectors expanded in a basis,

$$\begin{aligned}\mathbf{v} &= \sum_{i=1}^{3} v^i \hat{\mathbf{e}}_i \\ \mathbf{u} &= \sum_{i=1}^{3} u^i \hat{\mathbf{e}}_i\end{aligned}$$

we may take the linear combination,

$$\begin{aligned}\alpha\mathbf{u}+\beta\mathbf{v} &= \alpha\sum_{i=1}^{3} u^i\hat{\mathbf{e}}_i + \beta\sum_{i=1}^{3} v^i\hat{\mathbf{e}}_i \\ &= \alpha\left(u^1\hat{\mathbf{e}}_1+u^2\hat{\mathbf{e}}_2+u^3\hat{\mathbf{e}}_3\right)+\beta\left(v^1\hat{\mathbf{e}}_1+v^2\hat{\mathbf{e}}_2+v^3\hat{\mathbf{e}}_3\right) \\ &= \left(\alpha u^1\hat{\mathbf{e}}_1+\alpha u^2\hat{\mathbf{e}}_2+\alpha u^3\hat{\mathbf{e}}_3\right)+\left(\beta v^1\hat{\mathbf{e}}_1+\beta v^2\hat{\mathbf{e}}_2+\beta v^3\hat{\mathbf{e}}_3\right) \\ &= \alpha u^1\hat{\mathbf{e}}_1+\alpha u^2\hat{\mathbf{e}}_2+\alpha u^3\hat{\mathbf{e}}_3+\beta v^1\hat{\mathbf{e}}_1+\beta v^2\hat{\mathbf{e}}_2+\beta v^3\hat{\mathbf{e}}_3 \\ &= \alpha u^1\hat{\mathbf{e}}_1+\beta v^1\hat{\mathbf{e}}_1+\alpha u^2\hat{\mathbf{e}}_2+\beta v^2\hat{\mathbf{e}}_2+\alpha u^3\hat{\mathbf{e}}_3+\beta v^3\hat{\mathbf{e}}_3 \\ &= \left(\alpha u^1+\beta v^1\right)\hat{\mathbf{e}}_1+\left(\alpha u^2+\beta v^2\right)\hat{\mathbf{e}}_2+\left(\alpha u^3+\beta v^3\right)\hat{\mathbf{e}}_3 \\ &= \sum_{i=1}^{3}\left(\alpha u^i+\beta v^i\right)\hat{\mathbf{e}}_i\end{aligned}$$

Index notation lets us omit all of the intermediate steps in this calculation. Multiplication distributes over addition and addition commutes, so we can immediately see that

$$\alpha\sum_{i=1}^{3} u^i\hat{\mathbf{e}}_i+\beta\sum_{i=1}^{3} v^i\hat{\mathbf{e}}_i=\sum_{i=1}^{3}\left(\alpha u^i+\beta v^i\right)\hat{\mathbf{e}}_i$$

The inner product and cross product of two vectors is easy to accomplish. For the inner (dot) product,

$$\mathbf{u}\cdot\mathbf{v}=\left(\sum_{i=1}^{3} u^i\hat{\mathbf{e}}_i\right)\cdot\left(\sum_{j=1}^{3} v^j\hat{\mathbf{e}}_j\right)$$

Notice that in writing this expression, we have been careful to write different indices in the two sums. This way we always know which things are being summed with which. Distributing the dot product over the sum, and recalling that $(\alpha\mathbf{u})\cdot\mathbf{v}=\alpha\left(\mathbf{u}\cdot\mathbf{v}\right)$,

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \left(\sum_{i=1}^{3} u^i\hat{\mathbf{e}}_i\right)\cdot\left(\sum_{j=1}^{3} v^j\hat{\mathbf{e}}_j\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} u^iv^j\left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right)\end{aligned}$$

Something important has happened here. We started by multiplying two sums of three terms each, and end by writing a general expression, $u^iv^j\left(\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\right)$, that encompasses all of the resulting nine terms very concisely. Notice that we can bring both u^i and v^j anywhere in the expression because the i and j indices tell us that v^j is summed with $\hat{\mathbf{e}}_j$, no matter where they occur in the expression.

1.1 Dot product

Now we only need to know the dot products of the basis vectors. Since the vectors are orthonormal, we get 1 if $i = j$ and 0 if $i \neq j$. We write

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = g_{ij}$$

where in this basis g_{ij} is exactly this: 1 if $i = j$ and zero otherwise. This matrix is called the *metric*. In Euclidean space and Cartesian coordinates, the metric is just the unit matrix

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

but in a general basis it will differ from this. The metric is always symmetric, $g_{ij} = g_{ji}$. Introduce this symbol into our expression for the dot product,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \sum_{i=1}^{3} \sum_{j=1}^{3} u^i v^j \left(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} u^i v^j g_{ij} \end{aligned}$$

Now we introduce a further convention. When the metric is summed with the components of a vector, we get a related vector but with the index down:

$$v_i = \sum_{j=1}^{3} v^j g_{ij}$$

When g_{ij} is the unit matrix, then the three numbers v_i are the same as the the three numbers v^i, but this will not always be the case. This convention gives us the final form

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u^i v_i$$

Notice that the sum is still over one up and one down index.

We may define the inverse of raising an index by defining

$$g^{ij} \equiv \left(g_{ij} \right)^{-1}$$

The metric is the *only* object where the raised index version and lowered index versions are defined this way. For all other objects, the metric provides this relationship:

$$\begin{aligned} v^i &= \sum_{j=1}^{3} g^{ij} v_j \\ v_i &= \sum_{j=1}^{3} g_{ij} v^i \\ T_{ij} &= \sum_{m=1}^{3} \sum_{n=1}^{3} g_{im} g_{jn} T^{mn} \\ T^i_{\ j} &= \sum_{n=1}^{3} g_{jn} T^{in} \end{aligned}$$

The metric must not be confused with another important tensor, the Kronecker delta, which *always* has components 1 on the diagonal and 0 elsewhere, regardless of the basis. Furthermore, the Kronecker delta always has one raised and one lowered index,

$$\delta^i{}_j = \delta_j{}^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.2 Cross product

The cross product leads us to introduce another important object called the Levi-Civita tensor. In our Cartesian basis, it is a 3-index collection of numbers, ε_{ijk}. Such an object has 27 components, but the Levi-Civita tensor is defined to be totally antisymmetric – the sign changes under interchange of any pair of indices. Thus, for example, $\varepsilon_{123} = -\varepsilon_{132}$. This means that for any components with two of i, j or k having the same value, that component must vanish: we have $\varepsilon_{212} = -\varepsilon_{212}$ by interchanging the $2s$, and there for $\varepsilon_{212} = 0$. Most components vanish, the only nonvanishing ones being those where i, j, k are all different. The nonzero components all have value ± 1:

$$\begin{aligned} \varepsilon_{123} &= \varepsilon_{231} = \varepsilon_{312} = +1 \\ \varepsilon_{132} &= \varepsilon_{213} = \varepsilon_{321} = -1 \end{aligned}$$

Using ε_{ijk} we can write index expressions for the cross product and curl. Start by raising an index on ε_{ijk},

$$\varepsilon^i{}_{jk} = \sum_{m=1}^{3} \delta^{im} \varepsilon_{mjk}$$

Notice that when we have indices both up and down, we maintain the their horizontal displacement to keep track of which index is which. In this simple Cartesian case, $\varepsilon^i{}_{jk}$ has the same numerical values as ε_{ijk}. Now the i^{th} component of the cross product is given by

$$[\mathbf{u} \times \mathbf{v}]^i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^i{}_{jk} u^j v^k$$

It is crucial to distinguish between "free indices" – those which are not summed, and "dummy indices" which are summed over. The free indices in every term must match exactly; thus, our expression above has a single raised i index in both terms. The j, k indices are dummy indices, which must always occur in pairs, one up and one down.

We check this result by simply writing out the sums for each value of i,

$$\begin{aligned} [\mathbf{u} \times \mathbf{v}]^1 &= \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^1{}_{jk} u^j v^k \\ &= \varepsilon^1{}_{23} u^2 v^3 + \varepsilon^1{}_{32} u^3 v^2 + (\textit{all other terms are zero}) \\ &= u^2 v^3 - u^3 v^2 \\ [\mathbf{u} \times \mathbf{v}]^2 &= \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^2{}_{jk} u^j v^k \\ &= \varepsilon^2{}_{31} u^3 v^1 + \varepsilon^2{}_{13} u^1 v^3 \\ &= u^3 v^1 - u^1 v^3 \\ [\mathbf{u} \times \mathbf{v}]^3 &= \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon^3{}_{jk} u^j v^k \\ &= u^1 v^2 - u^2 v^1 \end{aligned}$$

We get the curl by replacing u^i by $\nabla_i = \frac{\partial}{\partial x^i}$, but the derivative operator is defined to have a down index, and this means we need to change the index positions on the Levi-Civita tensor again. Setting $\varepsilon^{ij}{}_k = \delta^{jm}\varepsilon^i{}_{mk}$ we have

$$[\nabla \times \mathbf{v}]^i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon^{ij}{}_k \nabla_j v^k$$

Checking individual components as above,

$$\begin{aligned}
[\nabla \times \mathbf{v}]^1 &= \frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3} \\
[\nabla \times \mathbf{v}]^2 &= \frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1} \\
[\nabla \times \mathbf{v}]^3 &= \frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}
\end{aligned}$$

If we sum these expressions with our basis vectors $\hat{\mathbf{e}}_i$, we may write these as vectors:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \sum_{i=1}^{3} [\mathbf{u} \times \mathbf{v}]^i \hat{\mathbf{e}}_i \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon^i{}_{jk} u^j v^k \hat{\mathbf{e}}_i \\
\nabla \times \mathbf{v} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon^{ij}{}_k \left(\nabla_j v^k\right) \hat{\mathbf{e}}_i
\end{aligned}$$

1.3 The Einstein summation convention

By now it is becoming evident that there are far too many summation symbols in our expressions. Fortunately, our insistence that dummy indices always occur in matched up-down pairs can releive us of this burden. Consider what happens to our expression for the cross product, for example, if we simply omit the summation symbols:

$$[\mathbf{u} \times \mathbf{v}]^i = \varepsilon^i{}_{jk} u^j v^k$$

There is a simple rule here: indices which occur matched across terms are free; indices which occur in matched up-down pairs are summed. We do not need the $\sum_{j=1}^{3}$ to see that the j indices should be summed. From now on, *every* matched, up-down pair of indices is to be summed. This is the *Einstein summation convention.*

Using the convention, our previous results are now written as:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= g_{ij} u^i v^j = u^i v_i \\
\mathbf{u} \times \mathbf{v} &= \varepsilon^i{}_{jk} u^j v^k \hat{\mathbf{e}}_i \\
\nabla \times \mathbf{v} &= \varepsilon^{ij}{}_k \left(\nabla_j v^k\right) \hat{\mathbf{e}}_i
\end{aligned}$$

and

$$\begin{aligned}
[\mathbf{u} \times \mathbf{v}]^i &= \varepsilon^i{}_{jk} u^j v^k \\
[\nabla \times \mathbf{v}]^i &= \varepsilon^{ij}{}_k \nabla_j v^k
\end{aligned}$$

1.4 Identities involving the Levi-Civita tensor

There are useful identities involving pairs of Levi-Civita tensors. The most general is

$$\varepsilon^{ijk}\varepsilon_{lmn} = \delta^i{}_l\delta^j{}_m\delta^k{}_n + \delta^i{}_m\delta^j{}_n\delta^k{}_l + \delta^i{}_n\delta^j{}_l\delta^k{}_m - \delta^i{}_l\delta^j{}_n\delta^k{}_m - \delta^i{}_n\delta^j{}_m\delta^k{}_l - \delta^i{}_m\delta^j{}_l\delta^k{}_n$$

To check this, first notice that the right side is antisymmetric in i, j, k and antisymmetric in l, m, n. For example, if we interchange i and j, we get

$$\varepsilon^{jik}\varepsilon_{lmn} = \delta^j{}_l\delta^i{}_m\delta^k{}_n + \delta^j{}_m\delta^i{}_n\delta^k{}_l + \delta^j{}_n\delta^i{}_l\delta^k{}_m - \delta^j{}_l\delta^i{}_n\delta^k{}_m - \delta^j{}_n\delta^i{}_m\delta^k{}_l - \delta^j{}_m\delta^i{}_l\delta^k{}_n$$

Now interchange the first pair of Kronecker deltas in each term, to get i, j, k in the original order, then rearrange terms, then pull out an overall sign,

$$\begin{aligned}\varepsilon^{jik}\varepsilon_{lmn} &= \delta^i{}_m\delta^j{}_l\delta^k{}_n + \delta^i{}_n\delta^j{}_m\delta^k{}_l + \delta^i{}_l\delta^j{}_n\delta^k{}_m - \delta^i{}_n\delta^j{}_l\delta^k{}_m - \delta^i{}_m\delta^j{}_n\delta^k{}_l - \delta^i{}_l\delta^j{}_m\delta^k{}_n \\ &= -\delta^i{}_l\delta^j{}_m\delta^k{}_n - \delta^i{}_m\delta^j{}_n\delta^k{}_l - \delta^i{}_n\delta^j{}_l\delta^k{}_m + \delta^i{}_l\delta^j{}_n\delta^k{}_m + \delta^i{}_n\delta^j{}_m\delta^k{}_l + \delta^i{}_m\delta^j{}_l\delta^k{}_n \\ &= -\varepsilon^{ijk}\varepsilon_{lmn}\end{aligned}$$

Total antisymmetry means that if we know one component, the others are all determined uniquely. Therefore, set $i = l = 1, j = m = 2, k = n = 3$, to see that

$$\begin{aligned}\varepsilon^{123}\varepsilon_{123} &= \delta^1{}_1\delta^2{}_2\delta^3{}_3 + \delta^1{}_2\delta^2{}_3\delta^3{}_1 + \delta^1{}_3\delta^2{}_1\delta^3{}_2 - \delta^1{}_1\delta^2{}_3\delta^3{}_2 - \delta^1{}_3\delta^2{}_2\delta^3{}_1 - \delta^1{}_2\delta^2{}_1\delta^3{}_3 \\ &= 1\cdot 1\cdot 1 + 0\cdot 0\cdot 0 + 0\cdot 0\cdot 0 - 1\cdot 0\cdot 0 - 0\cdot 1\cdot 0 - 0\cdot 0\cdot 1 \\ &= 1\end{aligned}$$

Check one more case. Let $i = 1, j = 2, k = 3$ again, but take $l = 3, m = 2, n = 1$. Then we have

$$\begin{aligned}\varepsilon^{123}\varepsilon_{321} &= \delta^1{}_3\delta^2{}_2\delta^3{}_1 + \delta^1{}_2\delta^2{}_1\delta^3{}_3 + \delta^1{}_1\delta^2{}_3\delta^3{}_2 - \delta^1{}_3\delta^2{}_1\delta^3{}_2 - \delta^1{}_1\delta^2{}_2\delta^3{}_3 - \delta^1{}_2\delta^2{}_3\delta^3{}_1 \\ &= 0\cdot 1\cdot 0 + 0\cdot 0\cdot 1 + 1\cdot 0\cdot 0 - 0\cdot 0\cdot 0 - 1\cdot 1\cdot 1 - 0\cdot 0\cdot 0 \\ &= -1\end{aligned}$$

as expected.

We get a second identity by setting $n = k$ and summing. Observe that the sum, $\delta^k{}_k = \delta^1{}_1 + \delta^2{}_2 + \delta^3{}_3 = 3$, while $\delta^j{}_k\delta^k{}_j = \delta^i{}_j$. We find

$$\begin{aligned}\varepsilon^{ijk}\varepsilon_{lmk} &= \delta^i{}_l\delta^j{}_m\delta^k{}_k + \delta^i{}_m\delta^j{}_k\delta^k{}_l + \delta^i{}_k\delta^j{}_l\delta^k{}_m - \delta^i{}_l\delta^j{}_k\delta^k{}_m - \delta^i{}_k\delta^j{}_m\delta^k{}_l - \delta^i{}_m\delta^j{}_l\delta^k{}_k \\ &= 3\delta^i{}_l\delta^j{}_m + \delta^i{}_m\delta^j{}_l + \delta^i{}_m\delta^j{}_l - \delta^i{}_l\delta^j{}_m - \delta^i{}_l\delta^j{}_m - 3\delta^i{}_m\delta^j{}_l \\ &= (3 - 1 - 1)\,\delta^i{}_l\delta^j{}_m - (3 - 1 - 1)\,\delta^i{}_m\delta^j{}_l \\ &= \delta^i{}_l\delta^j{}_m - \delta^i{}_m\delta^j{}_l\end{aligned}$$

so we have a much simpler, and very useful, relation

$$\varepsilon^{ijk}\varepsilon_{lmk} = \delta^i{}_l\delta^j{}_m - \delta^i{}_m\delta^j{}_l$$

A second sum gives another identity. Setting $m = j$ and summing again,

$$\begin{aligned}\varepsilon^{ijk}\varepsilon_{ljk} &= \delta^i{}_l\delta^j{}_j - \delta^i{}_j\delta^j{}_l \\ &= 3\delta^i{}_l - \delta^i{}_l \\ &= 2\delta^i{}_l\end{aligned}$$

Setting the last two indices equal and summing provides a check on our normalization,

$$\varepsilon^{ijk}\varepsilon_{ijk} = 2\delta^i{}_i = 6$$

This is correct, since there are only six nonzero components and we are summing their squares.

Collecting these results,

$$\begin{aligned}
\varepsilon^{jik}\varepsilon_{lmn} &= \delta^j_{\ l}\delta^i_{\ m}\delta^k_{\ n} + \delta^j_{\ m}\delta^i_{\ n}\delta^k_{\ l} + \delta^j_{\ n}\delta^i_{\ l}\delta^k_{\ m} - \delta^j_{\ l}\delta^i_{\ n}\delta^k_{\ m} - \delta^j_{\ n}\delta^i_{\ m}\delta^k_{\ l} - \delta^j_{\ m}\delta^i_{\ l}\delta^k_{\ n} \\
\varepsilon^{ijk}\varepsilon_{lmk} &= \delta^i_{\ l}\delta^j_{\ m} - \delta^i_{\ m}\delta^j_{\ l} \\
\varepsilon^{ijk}\varepsilon_{ljk} &= 2\delta^i_{\ l} \\
\varepsilon^{ijk}\varepsilon_{ijk} &= 6
\end{aligned}$$

We demonstrate the usefulness of these properties by proving some vector identities. First, consider the triple product. Working from the outside in,

$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= u^i\left[\mathbf{v}\times\mathbf{w}\right]_i \\
&= u^i\varepsilon_{ijk}v^j w^k \\
&= \varepsilon_{ijk}u^i v^j w^k
\end{aligned}$$

Because $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$, we may write this in two other ways,

$$\begin{aligned}
\varepsilon_{ijk}u^i v^j w^k &= \varepsilon_{jki}u^i v^j w^k = \varepsilon_{kij}u^i v^j w^k \\
u^i\varepsilon_{ijk}v^j w^k &= v^j\varepsilon_{jki}w^k u^i = w^k\varepsilon_{kij}u^i v^j \\
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})
\end{aligned}$$

proving that the triple product may be permuted cyclically.

Next, consider a double cross product:

$$\begin{aligned}
\left[\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\right]^i &= \varepsilon^i_{\ jk}u^j\left[\mathbf{v}\times\mathbf{w}\right]^k \\
&= \varepsilon^i_{\ jk}u^j\varepsilon^k_{\ lm}v^l w^m \\
&= \varepsilon^i_{\ jk}\varepsilon^k_{\ lm}u^j v^l w^m \\
&= \varepsilon^{ijk}\varepsilon_{lmk}u_j v^l w^m \\
&= \left(\delta^i_{\ l}\delta^j_{\ m} - \delta^i_{\ m}\delta^j_{\ l}\right)u_j v^l w^m \\
&= \delta^i_{\ l}\delta^j_{\ m}u_j v^l w^m - \delta^i_{\ m}\delta^j_{\ l}u_j v^l w^m \\
&= u_m v^i w^m - u_l v^l w^i \\
&= v^i\left(\mathbf{u}\cdot\mathbf{w}\right) - w^i\left(\mathbf{u}\cdot\mathbf{v}\right)
\end{aligned}$$

Returning fully to vector notation, this is the $BAC - CAB$ rule,

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \left(\mathbf{u}\cdot\mathbf{w}\right)\mathbf{v} - \left(\mathbf{u}\cdot\mathbf{v}\right)\mathbf{w}$$

Finally, look at the curl of a cross product,

$$\begin{aligned}
\left[\nabla\times(\mathbf{v}\times\mathbf{w})\right]^i &= \varepsilon^{ij}_{\ \ k}\nabla_j\left[\mathbf{v}\times\mathbf{w}\right]^k \\
&= \varepsilon^{ij}_{\ \ k}\nabla_j\left(\varepsilon^k_{\ lm}v^l w^m\right) \\
&= \varepsilon^{ij}_{\ \ k}\varepsilon^k_{\ lm}\nabla_j\left(v^l w^m\right) \\
&= \varepsilon^{ijk}\varepsilon_{klm}\nabla_j\left(v^l w^m\right) \\
&= \varepsilon^{ijk}\varepsilon_{lmk}\left(\left(\nabla_j v^l\right)w^m + v^l\left(\nabla_j w^m\right)\right) \\
&= \left(\delta^i_{\ l}\delta^j_{\ m} - \delta^i_{\ m}\delta^j_{\ l}\right)\left(\left(\nabla_j v^l\right)w^m + v^l\left(\nabla_j w^m\right)\right) \\
&= \delta^i_{\ l}\delta^j_{\ m}\left(\nabla_j v^l\right)w^m + \delta^i_{\ l}\delta^j_{\ m}v^l\left(\nabla_j w^m\right) - \delta^i_{\ m}\delta^j_{\ l}\left(\nabla_j v^l\right)w^m - \delta^i_{\ m}\delta^j_{\ l}v^l\left(\nabla_j w^m\right) \\
&= \left(\nabla_j v^i\right)w^j + v^i\left(\nabla_j w^j\right) - \left(\nabla_j v^j\right)w^i - v^j\nabla_j w^i
\end{aligned}$$

Restoring the vector notation, we have

$$\nabla \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla)\,\mathbf{v} + (\nabla \cdot \mathbf{w})\,\mathbf{v} - (\nabla \cdot \mathbf{v})\,\mathbf{w} - (\mathbf{v} \cdot \nabla)\,\mathbf{w}$$

If you doubt the advantages here, try to prove these identities by explicitly writing out all of the components!

1.5 Practice with index notation

Borrowed from Jackson, *Classical Electrodynamics*

1. Prove the following identities using index notation.

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}) \\
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})
\end{aligned}$$

2. The grad operator, ∇, is treated like a vector with components $\nabla_i = \frac{\partial}{\partial x_i}$, but it is also an operator. The thing to remember is that it always obeys the product rule. For example, for a function, f, and a vector, $\mathbf{a}$,

$$\begin{aligned}
\nabla \cdot (f\mathbf{a}) &= \nabla_i\,(f a_i) \\
&= (\nabla_i f)\,a_i + f \nabla_i a_i \\
&= (\nabla f) \cdot \mathbf{a} + f \nabla \cdot \mathbf{a}
\end{aligned}$$

   Prove the following two identities. Both of these require results involving symmetry:

$$\begin{aligned}
\nabla \times \nabla f &= 0 \\
\nabla \cdot (\nabla \times \mathbf{a}) &= 0
\end{aligned}$$

3. Prove the following identities:

$$\begin{aligned}
\nabla \times (f\mathbf{a}) &= (\nabla f) \times \mathbf{a} + f \nabla \times \mathbf{a} \\
\nabla \times (\nabla \times \mathbf{a}) &= \nabla\,(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \\
\nabla\,(\mathbf{a} \cdot \mathbf{b}) &= (\mathbf{a} \cdot \nabla)\,\mathbf{b} + (\mathbf{b} \cdot \nabla)\,\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \\
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}\,(\nabla \cdot \mathbf{b}) - \mathbf{b}\,(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\,\mathbf{a} - (\mathbf{a} \cdot \nabla)\,\mathbf{b}
\end{aligned}$$