Homework Problem: Spatially closed cosmology

Background

There are essentially 3 spatial geometries which are homogeneous and isotropic. The first, Euclidean space, we have investigated in class. Another is "hyperbolic space", which you can read about in your text (§12.2). Both of these spaces are infinite in extent. Here we briefly explore the third possibility: the universe is bounded but has no boundary — it is a 3-dimensional sphere. You have encountered the 3-sphere in a homework problem (Chapter 6, problem 33).

If, at each instant of time, the universe is homogeneous and isotropic with the geometry of a 3-sphere, then the spacetime metric can be put into the form

$$g = -dt \otimes dt + a^2(t) \left(d\chi \otimes d\chi + \sin^2 \chi \, d\theta \otimes d\theta + \sin^2 \chi \, \sin^2 \theta \, d\phi \otimes d\phi \right)$$

Here the scale factor $a(t)$ is the only unknown function. You can see that it determines a time-dependent "radius" for the 3-sphere. (Of course, the 3 sphere is not actually sitting inside a larger space.)

Your assignment

Your task is to use the *DifferentialGeometry* package in *Maple* to solve the Einstein equations for a metric of this form coupled to homogeneous and isotropic dust (vanishing pressure). In particular, show that the universe emerges from a singularity ("Big Bang"), expands for a while, contracts, and eventually collapses into a singularity ("Big Crunch").

Hint: It is not necessary, but some find the use of the conformal time coordinate η, defined by $d\eta = \frac{dt}{a(t)}$ (as discussed in class) to be more convenient for this cosmological model.

Submit to Torre before the last day of classes.