Relativistic Astrophysics
Assignment #1, Spring 2013

"Don't panic."
∼Douglas Adams

1: Effective Potentials [Maple Exercise]

In class, we showed how to cast the problem of motion in the Schwarzschild exterior in the context of an effective potential. Being able to work with these potentials is largely an exercise in being able to compute the appropriate values associated with a given set of physical parameters. This is most readily done by modern computational environments like *Maple* and *Mathematica*. Complete the following in the environment of your choice, and print out the notebook and attach it to your solutions. Please make sure your notebook has appropriate amounts of commentary so it is clear what each calculation is!

a ▶ Plot the effective potential for specific angular momentum $\ell/M = 4.5$ from $r/M = 0$ to $r/M = 70$.

b ▶ Numerically find the extrema of this potential by taking a derivative of your effective potential, and setting it equal to zero ($dV_{eff}/dR = 0$). At what values of r/M are V_{max} and V_{min}? What are the values of V_{max} and V_{min}?

c ▶ Plot $\mathcal{E} = -0.015$ on top of your plot of V_{eff}. Find the three turning points, r_I, r_p, and r_a.

HINT: You may find it convenient in Maple to work in a new variable, $R = r/M$.

2: Schwarzschild Vacuum Interior

a ▶ The conventional vacuum Schwarzschild metric is written in Schwarzschild coordinates as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$$

Make the transformation $T \equiv r$ and $R \equiv t$ in the regime where $r < 2M$. Write the metric in these new coordinates.

b ▶ In the exterior we had a timelike Killing vector $\xi^\alpha = \{1, 0, 0, 0\}$. This vector is timelike because $\xi^\alpha \xi_\alpha < 0$. Show that when $r < 2M$ this Killing vector is *spacelike.*

3: Stellar Remnants [∼ Schutz, 10.19]

Generically, ordinary stars like the Sun can be treated non-relativistically. However when the evolve into compact stellar remnants, relativistic treatments become more important.

As astrophysicists, we firmly believe in *Wheeler's First Moral Principle*[1]: you should always know the answer to a problem before you start. By this, we mean you should have some ballpark figure in your head about what the outcome of a problem might look like.

In the case of relativistic stars, we can make ballpark guesses about properties by conserving appropriate physical quantities when we shrink a star like the Sun to the appropriate size. Consider the Sun, which has an equatorial rotational speed of ~ 2 km/s.

a ▶ Estimate its angular momentum, on the assumption that the rotation is rigid (uniform angular velocity) and the Sun is of uniform density. As the true angular velocity is likely to increase inwards, this is the lower

[1] John Wheeler, not Jim Wheeler

limit on the Sun's angular momentum.

b ▶ If the Sun were to collapse to the size of a neutron star ($r = 10$ km), conserving both mass and total angular momentum, what would its angular velocity of rigid rotation be? In non-relativistic language, would the corresponding centrifugal force exceed the Newtonian gravitational force on the equator?

c ▶ A neutron star of $1M_{\odot}$ and radius 10km rotates 30 times per second (typical of young neutron stars). Again in Newtonian language, what is the ratio of centrifugal to gravitational force on the equator? In this sense, the star is slowly rotating.

d ▶ Suppose a main-sequence star of $1M_{\odot}$ has a dipole magnetic field with typical strength 1 G in the equatorial plane. Assuming flux conservation in this plane, what field strength should we expect if the star collapses to a radius of 10 km? (The Crab pulsar has a field $\sim 10^{11}$G.)

4: Astrophysical Densities

a ▶ Calculate the density of the Sun.

b ▶ Assume the supermassive black hole at the center of the galaxy has a Schwarzschild radius r_s and mass of $4 \times 10^6 M_{\odot}$. Calculate the "average density" of the black hole, defined as

$$\rho = \frac{M}{(4/3)\pi r_s^3}$$

c ▶ Compare your answers to part (a) and (b), and comment on what you think of this result.

d ▶ Using the average density formula in part (b), compute ρ for a black hole with mass $10M_{\odot}$. Again, compare your answers with (a) and (b), and comment on what this result means.