Symmetry

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1 The Lie Derivative

With or without the covariant derivative, which requires a connection on all of spacetime, there is another sort of derivation called the Lie derivative, which requires only a curve.

Let $C : R \rightarrow \mathcal{M}$ be a curve in $\mathcal{M}$ with tangent vectors, $\xi = \frac{d}{d\lambda}$, with components

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \frac{dx^\mu}{\partial \lambda} \frac{\partial}{\partial x^\mu}$$

The Lie derivative generalizes the directional derivative of a function,

$$\frac{df}{d\lambda} = \xi^\mu \frac{\partial f}{\partial x^\mu}$$

to higher rank tensors. First, consider a vector field, v, defined on $\mathcal{M}$. We define the Lie derivative of v at a point $\mathcal{P}$ along C to be

$$\mathscr{L}_\xi v = \lim_{\varepsilon \rightarrow 0} \frac{v(\mathcal{P} + \varepsilon \xi) - v(\mathcal{P})}{\varepsilon}$$

where $v(\mathcal{P} + \varepsilon \xi)$ is the Lie transport of v along the curve. For simplicity, let $\mathcal{P} = C(\lambda = 0)$. Lie transport involves taking the value of the vector field at a point on C, say, $v(\lambda)$, and performing a coordinate transformation to bring the point $C(\lambda)$ back to $\mathcal{P} = C(0)$. The coordinate transformation we require is, for infinitesimal $\lambda = \varepsilon$,

$$y^\alpha = x^\alpha - \varepsilon \xi^\alpha (0)$$

The components of v^α change as

$$\begin{aligned}
\tilde{v}^\alpha (0) &= v^\beta (\lambda) \frac{\partial y^\alpha}{\partial x^\beta} \\
&= \left[v^\beta \left(x^\mu (0) + \varepsilon \xi^\mu \right) \right] \left(\delta^\alpha_\beta - \varepsilon \partial_\beta \xi^\alpha \right) \\
&= \left[v^\beta (0) + \varepsilon \xi^\mu \partial_\mu v^\beta (0) \right] \left(\delta^\alpha_\beta - \varepsilon \partial_\beta \xi^\alpha \right) \\
&= v^\alpha (0) + \varepsilon \xi^\mu \partial_\mu v^\alpha (0) - \varepsilon v^\beta (0) \partial_\beta \xi^\alpha
\end{aligned}$$

The derivative is then

$$\begin{aligned}
\mathscr{L}_\xi v &= \lim_{\varepsilon \rightarrow 0} \frac{v(\mathcal{P} + \varepsilon \xi) - v(\mathcal{P})}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{v^\alpha (0) + \varepsilon \xi^\mu \partial_\mu v^\alpha (0) - \varepsilon v^\beta (0) \partial_\beta \xi^\alpha - v(0)}{\varepsilon} \\
&= \xi^\mu \partial_\mu v^\alpha (0) - v^\beta (0) \partial_\beta \xi^\alpha
\end{aligned}$$

An easy proof of the covariance of this result is that it equals the commutator of the two vectors,

$$\mathscr{L}_\xi v = [\xi, v]$$

which has the same form when ξ and v are expanded in components,

$$\begin{aligned}
[\xi, v] &= \left[\xi^\alpha \partial_\alpha, v^\beta \partial_\beta\right] \\
&= \xi^\alpha \partial_\alpha v^\beta \partial_\beta - v^\beta \partial_\beta \xi^\alpha \partial_\alpha \\
&= \left(\xi^\beta \partial_\beta v^\alpha - v^\beta \partial_\beta \xi^\alpha\right) \partial_\alpha
\end{aligned}$$

The generalization to higher rank tensors is immediate because derivations must satisfy the Leibnitz rule. Thus, for an outer product of two vectors,

$$T^{\alpha\beta} = u^\alpha v^\beta$$

we have

$$\begin{aligned}
\mathscr{L}_\xi T^{\alpha\beta} &= \mathscr{L}_\xi \left(u^\alpha v^\beta\right) \\
&= \left(\mathscr{L}_\xi u^\alpha\right) v^\beta + u^\alpha \left(\mathscr{L}_\xi v^\beta\right) \\
&= \left(\xi^\mu \partial_\mu u^\alpha - u^\mu \partial_\mu \xi^\alpha\right) v^\beta + u^\alpha \left(\xi^\mu \partial_\mu v^\beta - v^\mu \partial_\mu \xi^\beta\right) \\
&= \xi^\mu \partial_\mu \left(u^\alpha v^\beta\right) - u^\mu v^\beta \partial_\mu \xi^\alpha - u^\alpha v^\mu \partial_\mu \xi^\beta \\
&= \xi^\mu \partial_\mu T^{\alpha\beta} - T^{\mu\beta} \partial_\mu \xi^\alpha - T^{\alpha\mu} \partial_\mu \xi^\beta
\end{aligned}$$

and so on for higher ranks, with one correction term, $-T^{\alpha\cdots\mu\cdots\beta}\partial_\mu \xi^\nu$, for each index. For forms, we use the directional derivative of a scalar,

$$\mathscr{L}_\xi \phi = \xi^\mu \frac{\partial \phi}{\partial x^\mu}$$

together with $\phi = v^\alpha \omega_\alpha$, for arbitrary v^α,

$$\begin{aligned}
\xi^\mu \frac{\partial \left(v^\alpha \omega_\alpha\right)}{\partial x^\mu} &= \mathscr{L}_\xi \left(v^\alpha \omega_\alpha\right) \\
\xi^\mu \left(\partial_\mu v^\alpha\right) \omega_\alpha + v^\alpha \xi^\mu \partial_\mu \omega_\alpha &= \left(\mathscr{L}_\xi v^\alpha\right) \omega_\alpha + v^\alpha \mathscr{L}_\xi \omega_\alpha \\
\xi^\mu \left(\partial_\mu v^\alpha\right) \omega_\alpha + v^\alpha \xi^\mu \partial_\mu \omega_\alpha &= \xi^\beta \left(\partial_\beta v^\alpha\right) \omega_\alpha - v^\beta \left(\partial_\beta \xi^\alpha\right) \omega_\alpha + v^\alpha \mathscr{L}_\xi \omega_\alpha \\
v^\alpha \xi^\mu \partial_\mu \omega_\alpha &= -v^\beta \left(\partial_\beta \xi^\alpha\right) \omega_\alpha + v^\alpha \mathscr{L}_\xi \omega_\alpha \\
v^\alpha \mathscr{L}_\xi \omega_\alpha &= v^\alpha \xi^\mu \partial_\mu \omega_\alpha + v^\alpha \left(\partial_\alpha \xi^\beta\right) \omega_\beta
\end{aligned}$$

Since this must hold for all v^α,

$$\mathscr{L}_\xi \omega_\alpha = \xi^\mu \partial_\mu \omega_\alpha + \omega_\beta \partial_\alpha \xi^\beta$$

2 Symmetry

The Lie derivative is just the right tool for finding symmetry of a metric. The Lie derivative of the metric tensor is

$$\mathscr{L}_\xi g_{\alpha\beta} = \xi^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\beta} \partial_\alpha \xi^\mu + g_{\alpha\mu} \partial_\beta \xi^\mu$$

Now suppose we have a congruence of curves, so that the collected tangent vectors form a vector field. Suppose further we choose coordinates so that λ is one of the coordinates, x^{α_0}, for α_0 a single fixed direction. Then the components of ξ^μ are constant,

$$\begin{aligned}
\xi^\mu &= \frac{dx^\mu}{d\lambda} \\
&= \delta^\mu_{\alpha_0}
\end{aligned}$$

For example, if the curve is timelike, then we choose $t = \lambda$ and we have $\xi^\mu = (1, 0, 0, 0)$. For such a choice, $\partial_\beta \xi^\mu = 0$ and the Lie derivative of the metric is just

$$\begin{aligned}
\mathscr{L}_\xi g_{\alpha\beta} &= \frac{dx^\mu}{d\lambda} \partial_\mu g_{\alpha\beta} \\
&= \frac{\partial g_{\alpha\beta}}{\partial \lambda}
\end{aligned}$$

This means that if this Lie derivative vanishes, the metric is independent of the coordinate λ. Since the metric then does not change along the congruence of curves, we have a symmetry of the spacetime. Any direction in which the metric is not changing is called an isometry.

We can find a differential equation to describe such symmetry directions. Setting the Lie derivative of the metric to zero, we have

$$\begin{aligned}
0 &= \xi^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\beta}\partial_\alpha \xi^\mu + g_{\alpha\mu}\partial_\beta \xi^\mu \\
&= \xi^\mu \partial_\mu g_{\alpha\beta} + \left[\partial_\alpha \left(g_{\mu\beta}\xi^\mu\right) - \xi^\mu \partial_\alpha g_{\mu\beta}\right] + \left[\partial_\beta \left(g_{\alpha\mu}\xi^\mu\right) - \xi^\mu \partial_\beta g_{\alpha\mu}\right] \\
&= \xi^\mu \partial_\mu g_{\alpha\beta} + \partial_\alpha \xi_\beta - \xi^\mu \partial_\alpha g_{\mu\beta} + \partial_\beta \xi_\alpha - \xi^\mu \partial_\beta g_{\alpha\mu} \\
&= \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha - \xi^\mu \partial_\alpha g_{\mu\beta} - \xi^\mu \partial_\beta g_{\alpha\mu} + \xi^\mu \partial_\mu g_{\alpha\beta} \\
&= \partial_\alpha \xi_\beta - \xi^\mu \frac{1}{2}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right) + \partial_\beta \xi_\alpha - \xi^\mu \frac{1}{2}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right) \\
&= \partial_\alpha \xi_\beta - \xi^\mu \Gamma_{\mu\beta\alpha} + \partial_\beta \xi_\alpha - \xi^\mu \Gamma_{\mu\alpha\beta} \\
&= \partial_\alpha \xi_\beta - \xi_\mu \Gamma^\mu_{\ \beta\alpha} + \partial_\beta \xi_\alpha - \xi_\mu \Gamma^\mu_{\ \alpha\beta} \\
&= D_\alpha \xi_\beta + D_\beta \xi_\alpha
\end{aligned}$$

resulting in the Killing equation,

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$$

Given the metric, we can ask for all solutions to this equation. Solutions, if they exist, represent symmetry directions of the spacetime, i.e., directions in which the metric is unchanging.

3 Example: Symmetries of Minkowski spacetime

Consider flat spacetime, for which the metric is Minkowski, $\eta_{\mu\nu}$. In Cartesian coordinates,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and the Christoffel connection vanishes, $\Gamma^\alpha_{\ \mu\nu} = 0$. Then we may replace the covariant derivatives by partial derivatives, and the Killing equation is simply

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0$$

Taking a further derivative, we have

$$\xi_{\alpha,\beta\mu} + \xi_{\beta,\alpha\mu} = 0$$

Now, cycle the indices twice, to give

$$\begin{aligned}
\xi_{\alpha,\beta\mu} + \xi_{\beta,\alpha\mu} &= 0 \\
\xi_{\beta,\mu\alpha} + \xi_{\mu,\beta\alpha} &= 0 \\
\xi_{\mu,\alpha\beta} + \xi_{\alpha,\mu\beta} &= 0
\end{aligned}$$

Adding the first two and subtracting the third we find

$$\begin{aligned}
0 &= \xi_{\alpha,\beta\mu} + \xi_{\beta,\alpha\mu} + \xi_{\beta,\mu\alpha} + \xi_{\mu,\beta\alpha} - \xi_{\mu,\alpha\beta} - \xi_{\alpha,\mu\beta} \\
&= 2\xi_{\beta,\alpha\mu}
\end{aligned}$$

so that the second derivative of ξ_β vanishes. This means that ξ_β must be linear in the coordinates,

$$\xi_\alpha = a_\alpha + b_{\alpha\beta} x^\beta$$

Substituting this into the Killing equation,

$$\begin{aligned} 0 &= \xi_{\alpha,\beta} + \xi_{\beta,\alpha} \\ &= b_{\alpha\beta} + b_{\beta\alpha} \end{aligned}$$

so that a_α is arbitrary while $b_{\alpha b}$ must be antisymmetric.

We now have 10 independent vector fields, each of the form

$$\xi_\alpha = a_\alpha + b_{\alpha\beta} x^\beta$$

for independent choices of the 10 constants a_α and $b_{\alpha\beta} = -b_{\beta\alpha}$. The simplest choice of the 10 vector fields is to take only one of the constants nonzero. If we take $b_{\alpha\beta} = 0$ and one of the components (say, m for $m =, 0, 1, 2, 3$) of a_α nonzero, we get four constant vector fields,

$$\xi^\alpha_{(m)} = \delta^\alpha_m$$

This represents a unit vector in each of the coordinate directions. Since they are constant, the integral curves are just the Cartesian coordinate axes, and the metric is indeed independent of each of these.

Now setting $a_\alpha = 0$ and choosing one of the six antisymmetric matrices $b_{\alpha\beta}$, we have either rotations or boosts. For example, with $b_{21} = -b_{12} = 1$, with all the rest zero, the vector field is

$$\begin{aligned} \xi &= \xi^\alpha \partial_\alpha \\ &= \left(\eta^{\alpha\beta} b_{\beta\mu} x^\mu\right) \partial_\alpha \\ &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{aligned}$$

This is the generator of a rotation around the z axis. Similarly, $b_{23} = -b_{32}$ and $b_{31} = -b_{13}$ lead to the generators of rotations around the x and y axes. If one of the nonzero indices is time, then we have a boost because of the sign change. For $b_{10} = -b_{01} = 1$, we find

$$\begin{aligned} \xi &= \xi^\alpha \partial_\alpha \\ &= \left(\eta^{\alpha\beta} b_{\beta\mu} x^\mu\right) \partial_\alpha \\ &= x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \end{aligned}$$

This is a generator for a Lorentz transformation. To see this, exponentiate the generator with a parameter,

$$\begin{aligned} \Lambda &= \exp\left[\lambda\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right)\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right)^n \end{aligned}$$

Consider the effect on the coordinates (t, x, y, z). Clearly, $\Lambda y = \Lambda z = 0$. For t we need

$$\begin{aligned} \left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right) t &= x \\ \left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right)^2 t &= \left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right) x \\ &= t \end{aligned}$$

and so on, alternating between x and t. The even and odd parts of the series therefore sum separately,

$$\Lambda t = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right)^n t$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}\lambda^{2m+1}x + \sum_{m=0}^{\infty} \frac{1}{(2m)!}\lambda^{2m}t \\
&= x\sinh\lambda + t\cosh\lambda
\end{aligned}
$$

Similarly, acting on x we get

$$
\Lambda x = t\sinh\lambda + x\cosh\lambda
$$

We recognize λ as the rapidity, and the full transformation,

$$
\begin{aligned}
\Lambda t &= x\sinh\lambda + t\cosh\lambda \\
\Lambda x &= t\sinh\lambda + x\cosh\lambda \\
\Lambda y &= y \\
\Lambda z &= z
\end{aligned}
$$

as a boost in the x-direction.

We therefore find exactly 10 isometries in Minkowski space. This is the maximum number of independent solutions to the Killing equation. The static, spherically symmetric Schwarzschild solution had one timelike Killing field and three spatial rotational Killing fields for a total of three. A generic spacetime has no isometries.

4 Example: Static, Spherically Symmetric Spacetimes

We may now say what we mean by a static, spherically symmetric spacetime. To be static, there must be a timelike Killing vector field; to be spherically symmetric, we require a full set of three rotational (hence spacelike) Killing vectors.

We use the Lie derivative to say restrict the form of the metric for a static, spherically symmetric spacetime.

If we want a static spacetime, it means that we want there to exist a *timelike* Killing vector field. Choosing the time coordinate to be the parameter $t = \lambda$, the symmetry condition becomes

$$
\begin{aligned}
0 &= \mathcal{L}_\xi g_{\alpha\beta} \\
&= \xi^\mu \partial_\mu g_{\alpha\beta} + \partial_\alpha \xi^\mu g_{\mu\beta} + \partial_\beta \xi^\mu g_{\alpha\mu}
\end{aligned}
$$

However, with $x^0 = t = \lambda$, the components of ξ are constant, so that

$$
\partial_\alpha \xi^\mu = 0
$$

Therefore,

$$
\begin{aligned}
0 &= \xi^\mu \partial_\mu g_{\alpha\beta} \\
&= \frac{\partial}{\partial t}\left(g_{\alpha\beta}\right)
\end{aligned}
$$

and we have a coordinates system in which the metric is independent of the time coordinate.

For the spherical symmetry, we know that we have three rotational Killing vector fields which together generate $SO(3)$. We can pick two of these for coordinates, but they will not commute with one another, so the metric will not be independent of both coordinates. Starting with the familiar form

$$
\begin{aligned}
\xi_1 &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\
\xi_2 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\
\xi_3 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}
\end{aligned}
$$

it is natural to choose one coordinate, φ, such that

$$\frac{\partial}{\partial \varphi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is a Killing vector. To describe a second direction, we want a linear combination of the remaining two rotations,

$$\alpha(\varphi)\,\xi_1 + \beta(\varphi)\,\xi_2$$

and we want this to remain orthogonal to ξ_3,

$$\begin{aligned}
0 &= \langle \xi_3, \alpha\xi_1 + \beta\xi_2 \rangle \\
&= \left\langle x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle \\
&= x\left\langle \frac{\partial}{\partial y}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle - y\left\langle \frac{\partial}{\partial x}, \alpha\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \beta\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\rangle \\
&= x\left\langle \frac{\partial}{\partial y}, -\alpha z\frac{\partial}{\partial y}\right\rangle - y\left\langle \frac{\partial}{\partial x}, \beta z\frac{\partial}{\partial x}\right\rangle \\
&= -\alpha z x\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle - \beta z y\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle \\
&= -z(\alpha x + \beta y) \\
&= -r\sin\theta z(\alpha\cos\varphi + \beta\sin\varphi)
\end{aligned}$$

To get zero, we can take

$$\begin{aligned}
\alpha &= \sin\varphi \\
\beta &= -\cos\varphi
\end{aligned}$$

Then we have

$$\begin{aligned}
\xi_4 &= \xi_1\sin\varphi + \xi_2\cos\varphi \\
&= \sin\varphi\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) - \cos\varphi\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\
&= \sin\varphi\left(r\sin\theta\sin\varphi\frac{\partial}{\partial z} - r\cos\theta\frac{\partial}{\partial y}\right) - \cos\varphi\left(r\cos\theta\frac{\partial}{\partial x} - r\sin\theta\cos\varphi\frac{\partial}{\partial z}\right) \\
&= r\sin\theta\sin\varphi\sin\varphi\frac{\partial}{\partial z} - r\cos\theta\sin\varphi\frac{\partial}{\partial y} - r\cos\varphi\cos\theta\frac{\partial}{\partial x} + r\sin\theta\cos\varphi\cos\varphi\frac{\partial}{\partial z} \\
&= -\cos\theta\left(r\cos\varphi\frac{\partial}{\partial x} + r\sin\varphi\frac{\partial}{\partial y}\right) + r\sin\theta\frac{\partial}{\partial z} \\
&= -\cos\theta\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - r\sin\theta\frac{\partial}{\partial z}
\end{aligned}$$

Compare

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{x}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{xz}{r^2}\frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \varphi} \\
&= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial y} &= \frac{y}{r}\frac{\partial}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{yz}{r^2}\frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \varphi}
\end{aligned}$$

$$
\begin{aligned}
&= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial z} &= \frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial\theta} \\
&= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}
$$

so we have

$$
\begin{aligned}
\xi_4 &= -\cos\theta\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + r\sin\theta\frac{\partial}{\partial z} \\
&= \cos\theta\left(r\sin\theta\cos^2\varphi\frac{\partial}{\partial r} + \cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} + r\sin\theta\sin^2\varphi\frac{\partial}{\partial r} + \cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}\right) \\
&\quad -r\sin\theta\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= r\sin\theta\cos\theta\frac{\partial}{\partial r} + \cos^2\theta\frac{\partial}{\partial\theta} - r\sin\theta\cos\theta\frac{\partial}{\partial r} + \sin^2\theta\frac{\partial}{\partial\theta} \\
&= \cos^2\theta\frac{\partial}{\partial\theta} + \sin^2\theta\frac{\partial}{\partial\theta} \\
&= \frac{\partial}{\partial\theta}
\end{aligned}
$$

We may therefore take two of the Killing vectors to be

$$
\begin{aligned}
\xi_4 &= \frac{\partial}{\partial\theta} \\
\xi_3 &= \frac{\partial}{\partial\varphi}
\end{aligned}
$$

giving two coordinates, θ, φ, corresponding to symmetry directions. Since these do not commute, the metric cannot be independent of both.