General relativity

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We now have the Einstein equation in the form

$$G^{\alpha\beta} = \kappa T^{\alpha\beta}$$

where the Einstein tensor is given in terms of the Ricci tensor and Ricci scalar by

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$$

The solution of the Einstein equation for a static, spherically symmetric, vacuum spacetime is the Schwarzschild metric:

$$ds^2 = -\left(a - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{a - \frac{r_0}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

We still have to determine the constants in these equations. The value of κ in the Einstein equation tells us how strongly a given matter distribution affects the curvature of spacetime, while the constant r_0 in the Schwarzschild solution should be related to the mass of the spherical body that gives rise to it.

We now determine each of these constants by studying weak gravity situations.

1 Determining the constant in the Schwarzschild solution

We impose two conditions in order to determine the constants in the Schwarzschild solution:

1. At large distance from the spherical source, the spacetime becomes flat,

$$\lim_{r\to\infty} \left(ds^2\right) = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

2. For weak gravity and low velocity, the motion of a test particle should agree with the Newtonian prediction. Specifically, a geodesic of the Schwarzschild geometry, for $\frac{r_0}{r} \ll 1$ and $v \ll 1$ must agree with the Newtonian law,

$$-\frac{GMm}{r^2}\mathbf{r} = m\frac{d\mathbf{v}}{dt}$$

Satisfying the first condition immediately gives $a = 1$, in the line element. The second point requires us to look at the geodesic equation.

There is one remaining constant in this solution, and since we expect the solution to describe gravity near a spherical body, we should be able to determine its value by comparing this result with Newton's law of gravity. The comparison requires us to consider the motion of a particle in a region of small curvature and low velocity. It is sufficient to consider a particle orbit far from the source, where the orbit must look like the Newtonian result.

The motion of such a particle is given by the geodesic equation,

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\ \mu\nu} u^\mu u^\nu$$

where u^α is the 4-velocity of the particle. The 4-velocity is

$$\begin{aligned} u^\alpha &= \left(c\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) \\ &= \frac{dt}{d\tau}\left(c, \dot{r}, \dot{\theta}, \dot{\varphi}\right) \end{aligned}$$

where the dot denotes a t derivative, $\frac{d}{dt}$. The proper time is given by

$$\begin{aligned} d\tau^2 &= \left(1 - \frac{r_0}{r}\right) dt^2 + \frac{1}{c^2}\left(\frac{dr^2}{1 - \frac{r_0}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) \\ &= \left(1 - \frac{r_0}{r}\right) dt^2 + \frac{1}{1 - \frac{r_0}{r}}\frac{\dot{r}}{c^2} + \frac{r^2\dot{\theta}}{c^2} + \frac{r^2\dot{\varphi}^2 \sin^2\theta}{c^2} \end{aligned}$$

where the second line holds for low velocities. From this, we have

$$\frac{dt}{d\tau} = \left(1 - \frac{r_0}{r}\right)^{-1}$$

We need only the very lowest approximation to the geodesic equation. assuming $\frac{r_0}{r} \ll 1$ and $\frac{v^2}{c^2} \ll 1$, we have

$$\begin{aligned} \frac{dt}{d\tau} &\approx 1 \\ u^\alpha &\approx (c, 0, 0.0) \end{aligned}$$

This is enough to find the lowest order acceleration,

$$\begin{aligned} \frac{du^\alpha}{d\tau} &= -\Gamma^\alpha_{\ \mu\nu} u^\mu u^\nu \\ &= -c^2 \Gamma^\alpha_{\ 00} \end{aligned}$$

Consulting our table of connection coefficients, we see that the only contribution is from

$$\begin{aligned} \Gamma^r_{\ 00} &= \frac{ff'}{g^2} \\ &= f^3 f' \\ &= \left(1 - \frac{r_0}{r}\right)^{3/2} \frac{1}{2}\left(1 - \frac{r_0}{r}\right)^{-1/2}\left(\frac{r_0}{r^2}\right) \\ &\approx \frac{r_0}{2r^2} \end{aligned}$$

The only acceleration is therefore in the r direction, and is given by

$$\frac{du^r}{d\tau} = \frac{dt}{d\tau}\frac{d^2r}{dt^2} = -\frac{r_0 c^2}{2r^2}$$

so with $\frac{dt}{d\tau} \approx 1$

$$\frac{d^2r}{dt^2} = -\frac{r_0 c^2}{2r^2}$$

Comparing to the Newtonian prediction,

$$\begin{aligned} m\frac{d^2r}{dt^2} &= -\frac{GMm}{r^2} \\ \frac{d^2r}{dt^2} &= -\frac{GM}{r^2} \end{aligned}$$

we set

$$\begin{aligned}\frac{r_0 c^2}{2} &= GM \\ r_0 &= \frac{2GM}{c^2}\end{aligned}$$

We now have the Schwarzschild line element, given by

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Notice the importance of the cancellation of the mass of the particle in Newton's law:

$$\begin{aligned}m\frac{d^2 r}{dt^2} &= -\frac{GMm}{r^2} \\ \frac{d^2 r}{dt^2} &= -\frac{GM}{r^2}\end{aligned}$$

Without this, the acceleration would depend on the mass of the particle, and there would not be a single geometry that would account for all orbits. The line element is typically written choosing gravitational units,

$$\begin{aligned}G &= 1 \\ c &= 1\end{aligned}$$

so we have the more compact expression

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

This describes gravity for objects as diverse as the moon, neutron stars, and black holes.

2 The constant in the Einstein equation

To determine the constant in the Einstein equation, we compare it to the Newtonian gravity theory. While Newton's law of gravitation is often stated in terms of two point masses,

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

we can easily write it as a scalar field theory. First, write the potential,

$$\phi = -\frac{GM}{r}$$

and write the mass as an integral over a mass density, $\rho(\mathbf{x}')$. Then, sum over infinitesimal volume elements to get the total potential

$$\phi(\mathbf{x}) = \int d^3x' \frac{G\rho}{|\mathbf{x} - \mathbf{x}'|}$$

Then taking the Laplacian of both sides,

$$\begin{aligned}\nabla^2\phi(\mathbf{x}) &= \int d^3x' G\rho(\mathbf{x}') \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \\ &= -4\pi \int d^3x' G\rho(\mathbf{x}') \delta^3(\mathbf{x} - \mathbf{x}') \\ &= -4\pi G\rho(\mathbf{x})\end{aligned}$$

so we have

$$\nabla^2 \phi(\mathbf{x}) = -4\pi G \rho(\mathbf{x})$$

An important difference between general relativity and Newtonian gravity is that the potential and source differ. For Newtonian gravity, the potential and source are scalars, ϕ, ρ. But general relativity takes all forms of energy and momentum, $T^{\alpha\beta}$, into account for the source, with the equally diverse metric, $g_{\alpha\beta}$, providing the potential. To compare the two, we look only at the corresponding piece of the Einstein equation, $G^{00} = \kappa T^{00} = \kappa \rho c^2$. Because of the presence of c^2 in this term, it is the dominant term in weak gravity situations.

Now we need to develop the Einstein theory of gravity in a comparable form. This is achieved by linearizing the theory.

Consider a nearly flat metric of the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

where $h_{\alpha\beta} = h_{\beta\alpha}$ is small enough that we can ignore terms of order h^2. Then the inverse metric is given by

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$$

where

$$h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu}$$

Since we only keep terms to first order, we may use $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to raise and lower indices on any terms involving $h_{\mu\nu}$.

Since the connection must be at least linear in $h_{\alpha\beta}$, the connection-squared terms in the curvature may be neglected, giving the same form (for a different reason) as given in eq. 6.68:

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= \frac{1}{2}\left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}\right) \\ &= \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}\right) \end{aligned}$$

The Ricci tensor is then

$$\begin{aligned} R_{\beta\nu} &= g^{\alpha\mu} R_{\alpha\beta\mu\nu} \\ &= \frac{1}{2}\left(\eta^{\alpha\mu} + h^{\alpha\mu}\right)\left(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}\right) \\ &\approx \frac{1}{2}\eta^{\alpha\mu}\left(h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}\right) \\ &= \frac{1}{2}\left(h^{\alpha}_{\ \nu,\beta\alpha} - h_{,\beta\nu} + h^{\alpha}_{\ \beta,\nu\alpha} - \Box h_{\beta\nu}\right) \end{aligned}$$

where $\Box$ is the flat space d'Alembertian, $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. The Ricci scalar is

$$R = h^{\alpha\beta}{}_{,\beta\alpha} - \Box h$$

so the Einstein tensor is

$$G_{\beta\nu} = \frac{1}{2}\left(h^{\alpha}_{\ \nu,\beta\alpha} - h_{,\beta\nu} + h^{\alpha}_{\ \beta,\nu\alpha} - \Box h_{\beta\nu}\right) - \frac{1}{2}\eta_{\beta\nu}\left(h^{\alpha\mu}{}_{,\alpha\mu} - \Box h\right)$$

Let

$$\begin{aligned} H_{\alpha\beta} &= h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta} \\ H &= -h \\ h_{\alpha\beta} &= H_{\alpha\beta} - \frac{1}{2} H \eta_{\alpha\beta} \end{aligned}$$

Then

$$
\begin{aligned}
G_{\beta\nu} &= \frac{1}{2}\left(\left(H^{\alpha}_{\ \nu} - \frac{1}{2}H\delta^{\alpha}_{\nu}\right)_{,\beta\alpha} + H_{,\beta\nu} + \left(H^{\alpha}_{\ \beta} - \frac{1}{2}H\delta^{\alpha}_{\beta}\right)_{,\nu\alpha}\right) \\
&\quad -\frac{1}{2}\left(\Box\left(H_{\beta\nu} - \frac{1}{2}H\eta_{\beta\nu}\right) + \eta_{\beta\nu}\left(H^{\alpha\mu} - \frac{1}{2}H\eta^{\alpha\mu}\right)_{,\alpha\mu} + \eta_{\beta\nu}\Box H\right) \\
&= \frac{1}{2}\left(H^{\alpha}_{\ \nu,\beta\alpha} - \frac{1}{2}H_{,\beta\nu} + H_{,\beta\nu} + H^{\alpha}_{\ \beta,\nu\alpha} - \frac{1}{2}H_{,\beta\nu}\right) \\
&\quad -\frac{1}{2}\left(\Box H_{\beta\nu} - \frac{1}{2}\Box H\eta_{\beta\nu} + \eta_{\beta\nu}H^{\alpha\mu}{}_{,\alpha\mu} - \frac{1}{2}\eta_{\beta\nu}\Box H + \eta_{\beta\nu}\Box H\right) \\
&= \frac{1}{2}\left(H^{\alpha}_{\ \nu,\beta\alpha} + H^{\alpha}_{\ \beta,\nu\alpha} - \eta_{\beta\nu}H^{\alpha\mu}{}_{,\alpha\mu} - \Box H_{\beta\nu}\right)
\end{aligned}
$$

Now, if we perform an infinitesimal coordinate transformation,

$$x^{\alpha} \rightarrow y^{\alpha} = x^{\alpha} + \xi^{\alpha}$$

then the metric changes to

$$
\begin{aligned}
g_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}} &= g_{\mu\nu}\left(\delta^{\mu}_{\alpha} + \xi^{\mu}{}_{,\alpha}\right)\left(\delta^{\nu}_{\beta} + \xi^{\nu}{}_{,\beta}\right) \\
&= g_{\alpha\beta} + g_{\beta\mu}\xi^{\mu}{}_{,\alpha} + g_{\alpha\mu}\xi^{\mu}{}_{,\beta} + O\left(\xi^{2}\right)
\end{aligned}
$$

When we substitute our perturbative metric, this becomes

$$\eta_{\alpha\beta} + \tilde{h}_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta}$$

where we may choose ξ_{α} any way we like without changing the physics. Then $H_{\alpha\beta}$ changes by

$$
\begin{aligned}
\tilde{H}_{\alpha\beta} &= \tilde{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\tilde{h} \\
&= h_{\alpha\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta} - \frac{1}{2}\eta_{\alpha\beta}\left(h + 2\xi^{\mu}{}_{,\mu}\right) \\
&= H_{\alpha\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta} - \eta_{\alpha\beta}\xi^{\mu}{}_{,\mu}
\end{aligned}
$$

Choose ξ_{α} so that $\tilde{H}_{\alpha\beta}$ has vanishing divergence,

$$
\begin{aligned}
0 &= \tilde{H}_{\alpha\beta}{}^{,\beta} \\
&= H_{\alpha\beta}{}^{,\beta} + \xi_{\beta,\alpha}{}^{,\beta} + \xi_{\alpha,\beta}{}^{,\beta} - \xi^{\mu}{}_{,\mu\alpha} \\
&= H_{\alpha\beta}{}^{,\beta} + \xi_{\alpha,\beta}{}^{,\beta}
\end{aligned}
$$

by solving

$$\Box\xi_{\alpha} = -H_{\alpha\beta}{}^{,\beta}$$

This wave equation with source always has a solution for reasonable functions. Notice that this coordinate transformation still has a metric of the form $\eta_{\mu\nu} + \tilde{h}_{\mu\nu}$, so the curvature has the same form in terms of $\tilde{h}_{\mu\nu}$ as it does in terms of $h_{\mu\nu}$.

With this choice of coordinates, we have

$$\tilde{G}_{\beta\nu} = -\frac{1}{2}\Box\tilde{H}_{\beta\nu}$$

Let $\tilde{H}_{\alpha\beta}$ be static, so that the wave operator reduces to the Laplacian, and substitute this into the Einstein equation,

$$\nabla^2 \tilde{H}^{\alpha\beta} = -2\kappa T^{\alpha\beta}$$

The energy-momentum tensor is dominated by the T^{00} component,

$$T^{00} = \rho c^2$$

where ρ is the mass density. Therefore, we compare the 00 component to the Newtonian gravity equation. While the Einstein equation gives

$$\begin{aligned} -\frac{1}{2}\nabla^2 H^{00} &= \kappa T^{00} \\ &= \kappa c^2 \rho \end{aligned}$$

Newtonian gravity is described by

$$\nabla^2 \phi = -4\pi G \rho$$

Finally, we relate $\tilde{H}^{00}$ to the potential. From the Schwarzschild solution, we have the potential for a point mass in perturbative form:

$$\begin{aligned} \eta_{00} + h_{00} &= -1 + \frac{2GM}{rc^2} \\ h_{00} &= \frac{2GM}{rc^2} \\ &= \frac{2\phi}{c^2} \end{aligned}$$

To construct $\tilde{H}^{00}$, we need the trace as well. Write the full Schwarzschild line element as

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \\ &\approx -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 + \frac{2GM}{rc^2}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \\ &= ds_0^2 + \frac{2GM}{rc^2} dt^2 + \frac{2GM}{rc^2} dr^2 \end{aligned}$$

so that

$$h_{\mu\nu} = \begin{pmatrix} \frac{2GM}{rc^2} & & & \\ & \frac{2GM}{rc^2} & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

and therefore

$$\begin{aligned} h &= g^{\alpha\beta} h_{\alpha\beta} \\ &= 0 \end{aligned}$$

The complete $H_{\mu\nu}$ is therefore given by

$$H_{\mu\nu} = \begin{pmatrix} \frac{2GM}{rc^2} & & & \\ & \frac{2GM}{rc^2} & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

To get the divergence-free form of this, we require

$$\tilde{H}_{\alpha\beta} = H_{\alpha\beta} + \xi_{\beta,\alpha} + \xi_{\alpha,\beta} - \eta_{\alpha\beta}\xi^{\mu}_{\ ,\mu}$$

where

$$\begin{aligned}
\Box\xi_\alpha &= -H_{\alpha\beta}{}^{,\beta} \\
\Box\xi_0 &= -H_{00}{}^{,0} = 0 \\
\Box\xi_r &= -H_{rr}{}^{,r} \\
&= \frac{2GM}{r^2c^2}
\end{aligned}$$

The r-dependent part of the d'Alembertian turns this into

$$\begin{aligned}
\frac{1}{r}\frac{d^2}{dr^2}(r\xi_r) &= \frac{2GM}{r^2c^2} \\
\frac{d^2}{dr^2}(r\xi_r) &= \frac{2GM}{rc^2} \\
\frac{d}{dr}(r\xi_r) &= \frac{2GM}{c^2}\ln r \\
r\xi_r &= \frac{2GM}{c^2}(r\ln r - r) \\
\xi_r &= \frac{2GM}{c^2}(\ln r - 1)
\end{aligned}$$

The remaining components vanish, so we have

$$\begin{aligned}
\xi^{\mu}_{\ ,\mu} &= \xi^{r}_{\ ,r} \\
&= \frac{\partial}{\partial r}\left[\frac{2GM}{c^2}(\ln r - 1)\right] \\
&= \frac{2GM}{rc^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{H}_{00} &= H_{00} + \xi_{0,0} + \xi_{0,0} - \eta_{00}\xi^{\mu}_{\ ,\mu} \\
&= \frac{2GM}{rc^2} - \eta_{00}\frac{2GM}{rc^2} \\
&= \frac{4GM}{rc^2}
\end{aligned}$$

Therefore, in terms of the Newtonian potential,

$$\tilde{H}_{00} = -\frac{4\phi}{c^2}$$

so the Einstein equation gives

$$\begin{aligned}
-\frac{1}{2}\nabla^2\tilde{H}^{00} &= \kappa T^{00} \\
\frac{2}{c^2}\nabla^2\phi &= \kappa c^2\rho
\end{aligned}$$

Replacing $\nabla^2\phi$ with the Newtonian field equation, $\nabla^2\phi = -4\pi G\rho$, this becomes

$$\begin{aligned}
\frac{2}{c^2}(-4\pi G\rho) &= \kappa c^2\rho \\
\kappa &= -\frac{8\pi G}{c^4}
\end{aligned}$$

and the Einstein equation becomes

$$G^{\alpha\beta} = -\frac{8\pi G}{c^4} T^{\alpha\beta}$$

The relative sign depends on the sign in the definition of the curvature.