Parallel transport and curvature

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We now know how to take the derivative of tensors in a way that produces another tensor, allowing us to write equations that hold in any coordinate system (even in curved spaces). We know that scalars produced by contracting tensors are independent of coordinates and therefore give measurable physical quantities. We now use the covariant derivative to build a tensor that characterizes curvature.

Curvature may be defined as a measure of the infinitesimal amount a vector rotates when transported around an infinitesmal closed loop. We begin a geometric example of this, finding the radius of a 2-sphere using only "measurements" available from within the space. Then we will develop a precise notion of parallel transport of a vector, i.e., a way to move a vector along a curve without explicitly rotating it. We can then use this transport to examine the effect of moving a vector around a closed loop. This will give us a general form for the Riemann curvature tensor.

1 Curvature of the 2-sphere

While the 2-sphere may be viewed as the surface of a sphere embeded in Euclidean 3-space, we will make use only of distances and areas on the surface. Nonetheless, we can find the radius of the sphere, which provides a measure of the curvature of the surface. The curvature is larger when the sphere has smaller radius, with large spheres curving very slowly. We therefore expect the curvature to depend inversely on the radius.

We might define the curvature at a point of a 2-dimensional surface as a limit, using the observation that the relationship between the area of a region and the length of its boundary changes if the surface is curved. Unfortunately, this does not work For example, in the case of our 2-sphere, the surface area of the upper half of the sphere is $2\pi R^2$, while the circumference of the equator (the boundary of the upper half-sphere) has length $2\pi R$. Consider the ratio of circumference to area in the limit of small loops. A circle around the north pole at an angle of θ_0 from the pole has circumference $C = 2\pi R \sin\theta_0$ while the area enclosed may be found by integrating

$$\begin{aligned} A &= \int_0^{2\pi} R d\phi \int_0^{\theta_0} R \sin\theta d\theta \\ &= 2\pi R^2 \left(1 - \cos\theta_0\right) \end{aligned}$$

In the limit as $\theta_0 \to 0$, the ratio becomes

$$\begin{aligned} \lim_{\theta_0 \to 0} \frac{C}{A} &= \lim_{\theta_0 \to 0} \frac{2\pi R \sin\theta_0}{2\pi R^2 \left(1 - \cos\theta_0\right)} \\ &= \lim_{\theta_0 \to 0} \frac{\theta_0}{R\left(1 - \left(1 - \frac{1}{2}\theta_0^2\right)\right)} \\ &= \frac{2}{R} \lim_{\theta_0 \to 0} \frac{1}{\theta_0} \end{aligned}$$

which diverges. We could avoid the divergence by comparing C^2 to A, but then the result would not depend on R. We need to be more subtle.

Instead of circumference, consider the angular deficit. A circle in flat space has a circumference of $2\pi R$ where R is the radius of the circle and 2π the angle. If we define a circle in a small region of a curved 2-dimensional space to be the set of points at fixed distance r from a given point, then we can compute its arc-length, s. If we parameterize the curve by an angle, where $s = r\varphi$ then that angle progresses through an effective angle of $\varphi = \frac{s}{r}$ in one complete transit. We define the anglar deficit to be the difference between 2π and this φ,

$$\Delta = 2\pi - \frac{s}{r}$$

and we can find both s and r using the metric in the surface. Comparing the angular deficit to the area of the surface enclosed, we get a measure of curvature,

$$\begin{aligned} \mathcal{R} &= \lim_{A\to 0} \frac{\Delta}{A} \\ &= \lim_{A\to 0} \frac{2\pi - \frac{s}{r}}{A} \end{aligned}$$

We compute this for the 2-sphere. The line element is

$$ds^2 = R^2 d\theta + R^2 \sin^2\theta d\varphi^2$$

For a circle around the north pole of the sphere at an angle θ_0 from the pole, the radius is a line of constant φ, so $d\varphi = 0$ and

$$r = \int_0^{\theta_0} R d\theta = R\theta_0$$

while the arclength of the circle is given by integrating the constant θ curve,

$$\begin{aligned} s &= \int_0^{2\pi} R \sin\theta_0 d\varphi \\ &= 2\pi R \sin\theta_0 \end{aligned}$$

The angular deficit is

$$\Delta = 2\pi \left(1 - \frac{R\sin\theta_0}{R\theta_0}\right)$$

The area is found by integrating the area element $R^2 \sin\theta d\theta d\varphi$,

$$\begin{aligned} A &= \int_0^{2\pi} d\varphi \int_0^{\theta_0} d\theta R^2 \sin\theta \\ A &= 2\pi R^2 (1 - \cos\theta_0) \end{aligned}$$

The curvature is then

$$\begin{aligned} \mathcal{R} &= \lim_{A\to 0} \frac{2\pi\left(1 - \frac{1}{\theta_0}\sin\theta_0\right)}{2\pi R^2 (1-\cos\theta_0)} \\ &= \frac{1}{R^2} \lim_{\theta_0\to 0} \frac{1 - \frac{1}{\theta_0}\left(\theta_0 - \frac{1}{3!}\theta_0^3 + \cdots\right)}{\left(1 - \left(1 - \frac{1}{2}\theta_0^2 + \cdots\right)\right)} \\ &= \frac{1}{R^2} \lim_{\theta_0\to 0} \frac{\frac{1}{3!}\theta_0^2 + \cdots}{\frac{1}{2}\theta_0^2 + \cdots} \\ &= \frac{1}{3R^2} \end{aligned}$$

2 Gaussian curvature

Gauss developed a way to characterize cuvature for 2-dimensional spaces embeded in R^3, and showed that it can be computed without reference to the embedding.

We begin with a calculation that uses the embedding, but then show that we can compute the same result without reference to anything but the metric of the surface. To begin, consider the curvature of a curve $\mathbf{x}(\lambda) = (x(\lambda), y(\lambda))$ in the Euclidean plane and the parameter λ is arclength. The tangent vector to the curve is given by $\mathbf{n} = \frac{d\mathbf{x}}{d\lambda}$. This will always be a unit vector since we choose arclength as the parameter. The curvature of the curve is the magnitude of the rate of change of this unit normal,

$$\kappa = \left|\frac{d\mathbf{n}(\lambda)}{d\lambda}\right|$$

Notice that since $\mathbf{n} \cdot \mathbf{n} = 1$, differentiating shows that the rate of change of $\mathbf{n}$ is orthogonal to $\mathbf{n}$,

$$\mathbf{n} \cdot \frac{d\mathbf{n}}{d\lambda} = 0$$

so defining the unique unit vector orthogonal to be $\mathbf{m}$ we have

$$\frac{d\mathbf{n}}{d\lambda} = \kappa \mathbf{m}$$

We use this notion of curvature to define the Gaussian curvature of a 2-surface.

Exercise: Prove that the curvature of a circle, $\mathbf{x}(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right)$ is constant.

At any point, $\mathcal{P}$, of a surface, S, consider the normal, $\mathbf{n}$, to the surface. Choose any plane P containing this vector. Any two such planes are related by the angle, φ, between them, while they intersect in the line containing $\mathbf{n}$. We may therefore label all planes containing $\mathbf{n}$ by φ, giving $P(\varphi)$. The intersection of the surface S with any one of these planes will be a curve, $C(\varphi)$, lying in $P(\varphi)$. There is a unique circle in the plane $P(\varphi)$ which (a) passes through $\mathcal{P}$, (b) is tangent to $C(\varphi)$, and (c) has curvature $\kappa(\varphi)$ matching $C(\varphi)$. This is called the osculating (i.e., kissing) circle.

The curvatures of the full set of osculating circles give a bounded function, $\kappa(\varphi)$ on a bounded interval, $[0, 2\pi]$. The function therefore has a maximum and a minimum, κ_1 and κ_2 respectively called the principal curvatures. The curvature of the surface is defined as the product of the principal curvatures,

$$\mathcal{R} \equiv \kappa_1 \kappa_2$$

We compute the Gaussian curvature of the 2-sphere. It is easy to see that for the 2-sphere, the principal curvatures are equal to one another, since the intersection of a plane normal to the sphere always gives a great circle. Now consider a great circle through the north pole – any curve of constant φ will do. In the embedding 3-space, taking $\varphi = 0$, the curve is $x^i = (R\sin\theta, 0, R\cos\theta) = \left(R\sin\frac{s}{R}, 0, R\cos\frac{s}{R}\right)$, where s is the arclength. The unit tangent is therefore

$$\begin{aligned}
\mathbf{n} &= \frac{dx^i}{ds} \\
&= \left(\cos\frac{s}{R}, 0, -\sin\frac{s}{R}\right)
\end{aligned}$$

At the north pole, $s = 0$, and the tangent points in the x-direction as expected. The principle curvature, $\kappa_1 = \kappa_2$, is given by the magnitude of

$$\begin{aligned}
\frac{d\mathbf{n}}{ds} &= \frac{1}{R}\left(-\sin\frac{s}{R}, 0, -\cos\frac{s}{R}\right) \\
&= \frac{1}{R}\left(-\hat{\mathbf{k}}\right)
\end{aligned}$$

so we have $\kappa_1 = \kappa_2 = \frac{1}{R}$ and the Gaussian curvature is

$$\mathcal{R} = \frac{1}{R^2}$$

which differs by only a constant from our previous result.

3 Parallel transport

To define a general notion of curvature for an arbitrary space, we will need to use parallel transport to compare vectors at different positions on a manifold. As we have shown, parallel transport of a vector $\mathbf{v}$ along a curve with tangent $\mathbf{u}$ is given by solving

$$u^\alpha D_\alpha v^\beta = 0$$

We will also use the example of a 2-sphere, for which the metric

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

gives the connection components

$$\begin{aligned} \Gamma^\theta_{\ \varphi\varphi} &= -\sin\theta\cos\theta \\ \Gamma^\varphi_{\ \theta\varphi} = \Gamma^\varphi_{\ \varphi\theta} &= \frac{\cos\theta}{\sin\theta} \end{aligned}$$

and the solution for parallel transport around a curve at constant θ,

$$\begin{aligned} v^\theta(\varphi) &= A\cos\alpha\varphi + B\sin\alpha\varphi \\ v^\varphi(\varphi) &= C\cos\alpha\varphi + D\sin\alpha\varphi \end{aligned}$$

with the frequency α given by

$$\alpha = \cos\theta_0$$

4 Curvature

In Section 1, we found the curvature by taking the ratio of the angular deficit to the area enclosed, in the limit as the area shrinks to a point. However, to compute the angular deficit, we used the embedding of the 2-sphere in 3-space. This time, we will use the result of parallel transport to derive the result intrinsically, i.e., using only the metric. The first step is to show that the angular deficit is equal to the rotation of a parallely transported vector.

4.1 The angle of rotation produced by parallel transport

We have found the components of a general vector v^i on the sphere as it is transported around a circle at $\theta = \theta_0$. After completing one full circuit, the new components are $v^i(2\pi)$,

$$\begin{aligned} v^\theta(2\pi) &= v_0^\theta\cos(2\pi\cos\theta_0) + v_0^\varphi\sin\theta_0\sin(2\pi\cos\theta_0) \\ v^\varphi(2\pi) &= v_0^\varphi\cos(2\pi\cos\theta_0) - \frac{v_0^\theta}{\sin\theta_0}\sin(2\pi\cos\theta_0) \end{aligned}$$

This is rotated from the original vector, $v^i(0) = \left(v_0^\theta, v_0^\varphi\right)$ by some angle β. To find the angle, compute the inner product of $v^i(0)$ with $v^i(2\pi)$. Then dividing by the length of v^i give the cosine of the angle between the two vectors,

$$\cos\beta = \frac{g_{ij}v^i(0)\,v^j(2\pi)}{g_{ij}v^i(0)\,v^i(0)}$$

Recall that the length in the denomenator is unchanged by the parallel transport. It is given by

$$g_{ij}v^i(0)\,v^i(0) = \left(Rv_0^\theta\right)^2 + \left(R\sin\theta_0 v_0^\varphi\right)^2$$

The inner product in the numerator is

$$\begin{aligned}
g_{ij}v^{i}(0)\,v^{j}(2\pi) &= R^2\left(v_0^\theta v^\theta(2\pi)+v_0^\varphi v^\varphi(2\pi)\sin^2\theta_0\right)\\
&= R^2\left(v_0^\theta\left(v_0^\theta\cos(2\pi\cos\theta_0)+v_0^\varphi\sin\theta_0\sin(2\pi\cos\theta_0)\right)+v_0^\varphi\left(v_0^\varphi\cos(2\pi\cos\theta_0)-\frac{v_0^\theta}{\sin\theta_0}\sin(2\pi\cos\theta_0)\right)\sin^2\right.\\
&= R^2\left(\left(v_0^\theta\right)^2\cos(2\pi\cos\theta_0)+\left(v_0^\varphi\right)^2\sin^2\theta_0\cos(2\pi\cos\theta_0)\right)\\
&= R^2\left(\left(v_0^\theta\right)^2+\left(v_0^\varphi\right)^2\sin^2\theta_0\right)\cos(2\pi\cos\theta_0)
\end{aligned}$$

Taking the ratio, we have

$$\begin{aligned}
\cos\beta &= \frac{R^2\left(\left(v_0^\theta\right)^2+\left(v_0^\varphi\right)^2\sin^2\theta_0\right)\cos(2\pi\cos\theta_0)}{\left(Rv_0^\theta\right)^2+\left(R\sin\theta_0 v_0^\varphi\right)^2}\\
&= \cos(2\pi\cos\theta_0)
\end{aligned}$$

so that the angle between the two vectors is $\beta = 2\pi\cos\theta_0$. This means that the parallelly transported vector has rotated through a total angle

$$\begin{aligned}
\Delta &= 2\pi-\beta\\
&= 2\pi(1-\cos\theta_0)
\end{aligned}$$

This is the closely related to the angular deficit we found by looking at the embedding. The angular deficit was

$$\Delta = 2\pi\left(1-\frac{\sin\theta_0}{\theta_0}\right)$$

but if we think of $\frac{\sin\theta_0}{\theta_0}$ infinitesimally, $\frac{d\sin\theta_0}{d\theta_0}=\cos\theta_0$, the two agree. In any case, either is a valid measure of the rotational effect of the geometry, and the use of parallel transport is intrinsic to the space considered.

4.2 The area enclosed by the loop

The area enclosed by the loop is

$$\begin{aligned}
A &= R^2\int_0^{2\pi}d\varphi\int_0^{\theta_0}\sin\theta d\theta\\
&= -2\pi R^2\cos\theta|_0^{\theta_0}\\
&= 2\pi R^2(1-\cos\theta_0)
\end{aligned}$$

4.3 The curvature

The curvature is now given by the limit as we shrink the loop to a point,

$$\begin{aligned}
\mathcal{R} &= \lim_{\theta_0\to 0}\frac{\Delta}{A}\\
&= \lim_{\theta_0\to 0}\frac{2\pi(1-\cos\theta_0)}{2\pi R^2(1-\cos\theta_0)}\\
&= \frac{1}{R^2}
\end{aligned}$$

as we found in Section 1. This time, however, we used only the metric of the 2-sphere to find the answer.