Curvature singularity

We wish to show that there is a curvature singularity at $r = 0$ of the Schwarzschild solution. We cannot use either of the invariants$R$ or $R_{ab}R^{ab}$ since both the Ricci tensor and the Ricci scalar vanish. The next simplest invariant is $R_{abcd}R^{abcd}$

The curvatures are given by

$$\begin{aligned}
R^1_{010} &= \frac{1}{2}\nu_{,11}e^{\nu-\lambda} + \frac{1}{4}\nu_{,1}\nu_{,1}e^{\nu-\lambda} - \frac{1}{4}\nu_{,1}\lambda_{,1}e^{\nu-\lambda} \\
R^1_{212} &= \frac{1}{2}r\lambda_{,1}e^{-\lambda} \\
R^1_{313} &= \frac{r}{2}\lambda_{,1}e^{-\lambda}\sin^2\theta \\
R^2_{020} &= \frac{1}{2r}\nu_{,1}e^{\nu-\lambda} \\
R^2_{323} &= \left(1 - e^{-\lambda}\right)\sin^2\theta \\
R^3_{030} &= \frac{1}{2r}\nu_{,1}e^{\nu-\lambda}
\end{aligned}$$

where

$$\begin{aligned}
e^{\nu} &= 1 - \frac{2m}{r} = e^{-\lambda} \\
\nu_{,1} &= \frac{2m}{r^2\left(1 - \frac{2m}{r}\right)} = \frac{2m}{r\left(r - 2m\right)} \\
\lambda_{,1} &= -\frac{2m}{r\left(r - 2m\right)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
R^1_{010} &= \frac{1}{2}\left(\nu_{,11} + \nu_{,1}\nu_{,1}\right)e^{\nu-\lambda} \\
&= \frac{1}{2}\left(-\frac{4m}{r^3\left(1 - \frac{2m}{r}\right)} - \frac{4m^2}{r^4\left(1 - \frac{2m}{r}\right)^2} + \frac{4m^2}{r^4\left(1 - \frac{2m}{r}\right)^2}\right)\left(1 - \frac{2m}{r}\right)^2 \\
&= -\frac{2m}{r^3}\left(1 - \frac{2m}{r}\right) \\
R^1_{212} &= -\frac{m}{r} \\
R^1_{313} &= -\frac{m}{r}\sin^2\theta \\
R^2_{020} &= \frac{m}{r^3}\left(1 - \frac{2m}{r}\right) \\
R^2_{323} &= \frac{2m}{r}\sin^2\theta \\
R^3_{030} &= \frac{m}{r^3}\left(1 - \frac{2m}{r}\right)
\end{aligned}$$

Check

$$\begin{aligned}
R^1_{010} &= \frac{1}{2}\nu_{,11}e^{\nu-\lambda} + \frac{1}{4}\nu_{,1}\nu_{,1}e^{\nu-\lambda} - \frac{1}{4}\nu_{,1}\lambda_{,1}e^{\nu-\lambda} \\
&= \frac{1}{2}\left(\nu_{,11} + \nu_{,1}\nu_{,1}\right)e^{\nu-\lambda}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(-\frac{4m}{r^3\left(1-\frac{2m}{r}\right)} - \frac{4m^2}{r^4\left(1-\frac{2m}{r}\right)^2} + \frac{4m^2}{r^4\left(1-\frac{2m}{r}\right)^2}\right)\left(1-\frac{2m}{r}\right)^2 \\
&= -\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)
\end{aligned}
$$

Lowering the upper index,

$$
\begin{aligned}
R_{1010} &= -\frac{2m}{r^3} \\
R_{1212} &= -\frac{m}{r\left(1-\frac{2m}{r}\right)} \\
R_{1313} &= -\frac{m}{r\left(1-\frac{2m}{r}\right)}\sin^2\theta \\
R_{2020} &= \frac{m}{r}\left(1-\frac{2m}{r}\right) \\
R_{2323} &= 2mr\sin^2\theta \\
R_{3030} &= \frac{m}{r}\left(1-\frac{2m}{r}\right)\sin^2\theta
\end{aligned}
$$

and raising all four indices,

$$
\begin{aligned}
R^{1010} &= -\frac{2m}{r^3} \\
R^{1212} &= -\frac{m}{r^5}\left(1-\frac{2m}{r}\right) \\
R^{1313} &= -\frac{m}{r^5\sin^2\theta}\left(1-\frac{2m}{r}\right) \\
R^{2020} &= \frac{m}{r^5\left(1-\frac{2m}{r}\right)} \\
R^{2323} &= \frac{2m}{r^7\sin^2\theta} \\
R^{3030} &= \frac{m}{r^5\sin^2\theta\left(1-\frac{2m}{r}\right)}
\end{aligned}
$$

To compute the invariant, we will have sums including all rearrangements of the indices of the nonvanishing terms, for example,

$$
\begin{gathered}
R_{1212}R^{1212} \\
R_{2112}R^{2112} \\
R_{1221}R^{1221} \\
R_{2121}R^{2121}
\end{gathered}
$$

So each term must be counted four times. Therefore

$$
\begin{aligned}
R_{abcd}R^{abcd} &= 4\left(-\frac{2m}{r^3}\right)\left(-\frac{2m}{r^3}\right) \\
&\quad +4\left(-\frac{m}{r\left(1-\frac{2m}{r}\right)}\right)\left(-\frac{m}{r^5}\left(1-\frac{2m}{r}\right)\right) \\
&\quad +4\left(-\frac{m}{r\left(1-\frac{2m}{r}\right)}\sin^2\theta\right)\left(-\frac{m}{r^5\sin^2\theta}\left(1-\frac{2m}{r}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4m}{r}\left(1-\frac{2m}{r}\right)\frac{m}{r^5\left(1-\frac{2m}{r}\right)} \\
&+8mr\sin^2\theta\frac{2m}{r^7\sin^2\theta} \\
&+\frac{4m}{r}\left(1-\frac{2m}{r}\right)\sin^2\theta\frac{m}{r^5\sin^2\theta\left(1-\frac{2m}{r}\right)}
\end{aligned}
$$

so simplifying,

$$
\begin{aligned}
R_{abcd}R^{abcd} &= -\frac{16m^2}{r^6}+\frac{4m^2}{r^6}+\frac{4m^2}{r^6}+\frac{4m^2}{r^6}+\frac{16m^2}{r^6}+\frac{4m^2}{r^6} \\
&= \frac{16m^2}{r^6}
\end{aligned}
$$

which diverges strongly at $r=0$, but is regular at $r=2m$.

Regularity at r = 2m

It turns out that the Schwarzschild metric has only a coordinate singularity at $r=2m$. A change of coordinates removes the singular factor, $1-\frac{2m}{r}$. Null coordinates, u and v, such that holding either u or v constant gives a null geodesic, turn out to not only remove the singularity, but also give a clearer picture of the causal relationships near the star.

First, we find the null geodesics in the rt-plane. We have already shown that

$$
u^0=\frac{k}{1-\frac{2m}{r}}
$$

and the line element then tells us that

$$
\begin{aligned}
0 &= -\left(1-\frac{2m}{r}\right)\left(u^0\right)^2+\frac{\left(u^1\right)^2}{1-\frac{2m}{r}} \\
&= -\frac{k^2}{1-\frac{2m}{r}}+\frac{\left(u^1\right)^2}{1-\frac{2m}{r}}
\end{aligned}
$$

so that

$$
u^1=\pm k
$$

Taking the quotient,

$$
\frac{dr}{dt}=\frac{u^1}{u^0}=\pm\left(1-\frac{2m}{r}\right)
$$

and integrating,

$$
\begin{aligned}
\pm t &= \int\frac{dr}{1-\frac{2m}{r}} \\
&= \int\frac{rdr}{r-2m} \\
&= \int\frac{(r-2m)\,dr}{r-2m}+\int\frac{2mdr}{r-2m} \\
\pm t &= r+2m\ln(r-2m)+c
\end{aligned}
$$

Therefore, defining

$$
\begin{aligned}
u &= t + r + 2m\ln(r-2m) \\
v &= t - r - 2m\ln(r-2m)
\end{aligned}
$$

we see that $u = constant$ gives an ingoing null geodesic and $v = constant$ gives an outgoing null geodesic.

To write the line element in terms of u and v, compute their differentials,

$$
\begin{aligned}
du &= dt + dr\left(1 + \frac{2m}{r-2m}\right) \\
&= dt + \frac{dr}{1-\frac{2m}{r}} \\
dv &= dt - \frac{dr}{1-\frac{2m}{r}}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
dudv &= dt^2 - \frac{dr^2}{\left(1-\frac{2m}{r}\right)^2} \\
ds^2 &= -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2 \\
&= -\left(1-\frac{2m}{r}\right)dudv + r^2 d\Omega^2
\end{aligned}
$$

This is still singular at $r = 2m$, but now define

$$
\begin{aligned}
U &= e^{\frac{u}{4m}} \\
V &= e^{-\frac{v}{4m}}
\end{aligned}
$$

Then

$$
\begin{aligned}
dUdV &= -\frac{1}{16m^2}dudve^{\frac{u-v}{4m}} \\
&= -\frac{1}{16m^2}dudve^{\frac{r}{2m}+\ln(r-2m)} \\
&= -\frac{r}{16m^2}\left(1-\frac{2m}{r}\right)e^{\frac{r}{2m}}dudv
\end{aligned}
$$

Substituting into the line element, we have

$$
ds^2 = \frac{16m^2}{r}e^{-\frac{r}{2m}}dUdV + r^2 d\Omega^2
$$

where $r = r(U,V)$. There is clearly no problem at $r = 2m$, which is called the event horizon.

Radial infall

We now consider a particle falling into the black hole. Since the neighborhood of $r = 2m$ is now established to be regular, there can be nothing to keep a particle from falling across the horizon. We therefore consider a particle falling from just inside the horizon toward the singularity.

In this region, r becomes the timelike coordinate. The geodesic equation for u^0 may be written as

$$
\left(\frac{2m}{r}-1\right)u^0 = k
$$

with initial value at r_0 given by

$$\left(\frac{2m}{r_0}-1\right)u_0^0 \quad = \quad k$$

Since r is now the timelike coordinate, we may choose $u_0^0=0$, and therefore, $k=0$, and $u^0=0$ along the entire geodesic. Then u^1 is given by

$$\begin{aligned} -1 &= -\frac{1}{\left(\frac{2m}{r}-1\right)}\left(u^1\right)^2 \\ u^1 &= \sqrt{\frac{2m}{r}-1} \end{aligned}$$

Integrating,

$$\begin{aligned} \tau &= \int d\tau = -\int_{r_0}^{0}\frac{dr}{\sqrt{\frac{2m}{r}-1}} \\ &= -\int_{r_0}^{0}\frac{\sqrt{r}dr}{\sqrt{2m-r}} \end{aligned}$$

Letting $y=\sqrt{2m-r}$, and therefore, $r=2m-y^2$,

$$\tau \quad = \quad \frac{1}{2}\int\sqrt{2m-y^2}dy$$

Now let $y=\sqrt{2m}\sin\theta$ so that

$$\begin{aligned} \tau &= m\int_{r_0}^{0}\cos^2\theta d\theta \\ &= \frac{m}{2}\int_{r_0}^{0}\left(1+\cos 2\theta\right)d\theta \\ &= \frac{m}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{r_0}^{0} \\ &= \frac{m}{2}\left(\theta+\sin\theta\cos\theta\right)\Big|_{r_0}^{0} \\ &= \frac{m}{2}\left(\arcsin\sqrt{1-\frac{r}{2m}}+\sqrt{1-\frac{r}{2m}}\sqrt{\frac{r}{2m}}\right)\Bigg|_{r_0}^{0} \\ &= \frac{m}{2}\left(\frac{\pi}{2}-\arcsin\sqrt{1-\frac{r_0}{2m}}+\sqrt{1-\frac{r_0}{2m}}\sqrt{\frac{r_0}{2m}}\right) \end{aligned}$$

We may take the initial position to be $r_0=2m$, which gives

$$\tau \quad = \quad \frac{m\pi}{4}$$

which is, in particular, finite.

The infall doesn't take long for stellar sized black holes. For a black hole with 10 times the mass of the sun, we have

$$\begin{aligned} \tau &= \frac{GM\pi}{4c^2} \\ &= \frac{6.67\times10^{-11}\times1.99\times10^{31}\times\pi}{4\times9\times10^{16}} \\ &= 3690 \text{ sec} \end{aligned}$$

or just over an hour. For a black hole at the center of a galaxy, which may have a mass a million times as great, the infall will take just short of two years.

The escape of light

Finally, consider light which starts near the event horizon at $t = 0$. How long does it take to escape from the region of the black hole? We consider an outgoing null geodesic, which has v constant,

$$c = t - r - 2m \ln (r - 2m)$$

Suppose light leaves $r_0 = 2m + \delta$ at time $t = 0$. Then

$$c = -2m - \delta - 2m \ln \delta$$

and the light reaches radius r at time

$$ct = r + 2m \ln (r - 2m) - 2m - \delta - 2m \ln \delta$$

Compare this time to the time it would take in free space for light to travel from $2m + \delta$ to r, given by

$$t_0 = \frac{r - 2m - \delta}{c}$$

We find

$$\begin{aligned} \frac{t}{t_0} &= \frac{r + 2m \ln (r - 2m) - 2m - \delta - 2m \ln \delta}{r - 2m - \delta} \\ &= 1 + \frac{2m \ln (r - 2m) - 2m \ln \delta}{r - 2m - \delta} \\ &= 1 + \frac{2m}{r - 2m - \delta} \ln \frac{r - 2m}{\delta} \end{aligned}$$

Consider a black hole with 10 times the mass of the sun, so that

$$2m = \frac{2GM}{c^2} = 2950m$$

The time to reach the orbit of Earth $(1.5 \times 10^{11} m)$ from near the horizon would normally be about 500 seconds. This is increased by

$$\begin{aligned} \Delta t &= 500 \times \frac{2m}{r - 2m - \delta} \ln \frac{r - 2m}{\delta} \\ &= 500 \times \frac{2950}{1.5 \times 10^{11}} \ln \frac{1.5 \times 10^{11}}{\delta} \\ &= 9.83 \times 10^{-6} \ln \frac{1.5 \times 10^{11}}{\delta} \end{aligned}$$

For a distance of $\delta = 1cm$ above the horizon, the time delay is

$$\begin{aligned} \Delta t &= 9.83 \times 10^{-6} \ln \left(1.5 \times 10^{13}\right) \\ &= 2.98 \times 10^{-4} \,\text{sec} \end{aligned}$$

so a collapsing star will appear to settle to its Schwarzschild radius extremely quickly.