Radial geodesics in Schwarzschild spacetime

Spherically symmetric solutions to the Einstein equation take the form

$$ds^2 = -\left(1+\frac{a}{r}\right)dt^2 + \frac{dr^2}{1+\frac{a}{r}} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$$

where a is constant.

We also have the connection components, which now take the form (using $e^\nu = e^{-\lambda} = 1+\frac{a}{r}$, and therefore $\nu = \ln\left(1+\frac{a}{r}\right)$ and $\nu_{,1} = -\frac{a}{r^2}e^{-\nu}$

$$\begin{aligned}
\Gamma^0_{00} &= 0\\
\Gamma^0_{01} &= \Gamma^0_{10} = -\frac{a}{2r^2\left(1+\frac{a}{r}\right)}\\
\Gamma^1_{00} &= -\frac{1}{2}\left(1+\frac{a}{r}\right)\frac{a}{r^2}\\
\Gamma^1_{11} &= \frac{a}{2r^2\left(1+\frac{a}{r}\right)}\\
\Gamma^1_{01} &= \Gamma^1_{10} = 0\\
\Gamma^0_{11} &= 0\\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r}\\
\Gamma^1_{22} &= -r\left(1+\frac{a}{r}\right)\\
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r}\\
\Gamma^1_{33} &= -\left(1+\frac{a}{r}\right)r\sin^2\theta\\
\Gamma^3_{23} &= \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta}\\
\Gamma^2_{33} &= -\sin\theta\cos\theta
\end{aligned}$$

Now consider the geodesic equation for a particle which starts from rest at time $\tau = t = 0$.

$$0 = \frac{du^a}{d\tau} + \Gamma^a_{bc}u^b u^c$$

It might seem that the initial velocity 4-vector is $u^a = (1,0,0,0)$, but this is not allowed. Using the line element, $ds^2 = -d\tau^2$, we must have

$$-1 = -\left(1+\frac{a}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2\left(1+\frac{a}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{c^2}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{r^2}{c^2}\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

where c is the speed of light. If there is no initial motion in the spatial directions we must have

$$u^0_0 = \left(\frac{dt}{d\tau}\right)_0 = \frac{1}{\sqrt{1+\frac{a}{r_0}}}$$

where r_0 is the initial radial position.

The geodesic equation for each component is then (at the initial time),

$$0 = \frac{1}{c}\frac{du^0}{d\tau} + 2\Gamma^0_{01}u^0u^1$$

$$\begin{aligned}
0 &= \frac{1}{c}\frac{du^1}{d\tau} + \Gamma^1_{00}u^0u^0 + \Gamma^1_{11}u^1u^1 + \Gamma^1_{22}u^2u^2 + \Gamma^1_{33}u^3u^3 \\
0 &= \frac{1}{c}\frac{du^2}{d\tau} + 2\Gamma^2_{21}u^2u^1 + \Gamma^2_{33}u^3u^3 \\
0 &= \frac{1}{c}\frac{du^3}{d\tau} + 2\Gamma^3_{13}u^1u^3 + \Gamma^3_{23}u^2u^3
\end{aligned}$$

Notice that if $u^2 = 0$ or $u^3 = 0$ then the corresponding accelerations also vanish, so they remain zero. However, u^1 cannot remain zero.

For radial motion, substituting for the connection coefficients, we therefore have

$$\begin{aligned}
0 &= \frac{1}{c}\frac{du^0}{d\tau} - \frac{a}{r^2\left(1+\frac{a}{r}\right)}u^0u^1 \\
0 &= \frac{1}{c}\frac{du^1}{d\tau} - \left(1+\frac{a}{r}\right)\frac{a}{2r^2}u^0u^0 + \frac{a}{2r^2\left(1+\frac{a}{r}\right)}u^1u^1
\end{aligned}$$

We also have the relation given by the line element,

$$-1 = -\left(1+\frac{a}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1+\frac{a}{r}}\left(\frac{1}{c}\frac{dr}{d\tau}\right)^2$$

First equation

Integrate the first equation,

$$\begin{aligned}
0 &= \frac{du^0}{u^0} - \frac{adr}{r^2\left(1+\frac{a}{r}\right)} \\
&= \ln u^0 - \int \frac{adr}{r(r+a)} \\
&= \ln u^0 - \int \left(\frac{1}{r} - \frac{1}{r+a}\right) dr \\
&= \ln u^0 - \ln r + \ln(r+a) - \ln b
\end{aligned}$$

for some constant b, and therefore

$$\left(1+\frac{a}{r}\right)u^0 = b$$

Evaluating at the initial condition, we see that $b = \sqrt{1+\frac{a}{r_0}}$.

Radial component of velocity from the line element

Now substitute into the line element equation to find u^1,

$$\begin{aligned}
-1 &= -\left(1+\frac{a}{r}\right)\left(u^0\right)^2 + \frac{1}{1+\frac{a}{r}}\left(u^1\right)^2 \\
-1 &= -\left(1+\frac{a}{r}\right)\left(\frac{b}{1+\frac{a}{r}}\right)^2 + \frac{1}{1+\frac{a}{r}}\left(u^1\right)^2 \\
\left(u^1\right)^2 &= b^2 - \left(1+\frac{a}{r}\right) \\
u^1 &= \sqrt{b^2 - \left(1+\frac{a}{r}\right)} \\
&= \sqrt{\frac{a}{r_0} - \frac{a}{r}}
\end{aligned}$$

Now integrate to find $r(\tau)$,

$$\begin{aligned} c\tau &= \int \frac{dr}{\sqrt{\frac{a}{r_0} - \frac{a}{r}}} \\ &= \frac{1}{\sqrt{a}} \int \frac{\sqrt{r}dr}{\sqrt{\frac{r}{r_0} - 1}} \end{aligned}$$

Let $y = \sqrt{r}$. Then $dy = \frac{1}{2}\frac{dr}{\sqrt{r}}$, so

$$c\tau = \frac{2}{\sqrt{a}} \int \frac{y^2 dy}{\sqrt{\frac{y^2}{r_0} - 1}}$$

Now let $y = \sqrt{r_0}\cosh\xi$ so that

$$\begin{aligned} c\tau &= \frac{2}{\sqrt{a}} \int \frac{r_0 \cosh^2 \xi \sqrt{r_0} \sinh \xi d\xi}{\sinh \xi} \\ &= \frac{2 (r_0)^{3/2}}{\sqrt{a}} \int \cosh^2 \xi d\xi \\ &= \frac{(r_0)^{3/2}}{\sqrt{a}} \int \left(\cosh^2 \xi + 1 + \sinh^2 \xi\right) d\xi \\ &= \frac{(r_0)^{3/2}}{\sqrt{a}} \int (\cosh 2\xi + 1)\, d\xi \\ &= \frac{(r_0)^{3/2}}{\sqrt{a}} \left(\frac{1}{2}\sinh 2\xi + \xi\right) \\ &= \frac{(r_0)^{3/2}}{\sqrt{a}} \left(\sinh \xi \cosh \xi + \cosh^{-1} \sqrt{\frac{r}{r_0}}\right) \\ &= \frac{(r_0)^{3/2}}{\sqrt{a}} \left(\sqrt{\frac{r}{r_0}}\sqrt{\frac{r}{r_0} - 1} + \cosh^{-1} \sqrt{\frac{r}{r_0}}\right) \\ &= \frac{\sqrt{r_0}}{\sqrt{a}} r + \frac{(r_0)^{3/2}}{\sqrt{a}} \cosh^{-1} \sqrt{\frac{r}{r_0}} \end{aligned}$$

Second equation

We can also get this result from the second equation,

$$0 = \frac{1}{c}\frac{du^1}{d\tau} - \left(1 + \frac{a}{r}\right) \frac{a}{2r^2} u^0 u^0 + \frac{a}{2r^2\left(1 + \frac{a}{r}\right)} u^1 u^1$$

Substituting for u^0,

$$\begin{aligned} 0 &= \frac{1}{c}\frac{du^1}{d\tau} - \frac{1}{2r^2}\frac{ab^2}{1 + \frac{a}{r}} + \frac{a}{2r^2\left(1 + \frac{a}{r}\right)} u^1 u^1 \\ \frac{1}{c}\frac{du^1}{d\tau} &= \frac{1}{r^2}\frac{a}{1 + \frac{a}{r}} \left(\frac{1}{2}b^2 - \frac{1}{2}u^1 u^1\right) \end{aligned}$$

Then, bringing the u^1 dependent factor to the left and multiplying both sides by $u^1 = \frac{1}{c}\frac{dr}{d\tau}$,

$$\frac{u^1 du^1}{\frac{1}{2}b^2 - \frac{1}{2}u^1 u^1} = \frac{1}{r^2}\frac{a}{1 + \frac{a}{r}} dr$$

we can integrate,

$$\begin{aligned}
\int \frac{u^1 du^1}{b^2 - \frac{1}{2}u^1 u^1} &= \ln r - \ln(r+a) \\
\ln r - \ln(r+a) &= \int \frac{u^1 du^1}{\frac{1}{2}b^2 - \frac{1}{2}u^1 u^1} \\
&= \int \frac{dy}{b^2 - y} \\
&= -\ln\left(b^2 - y\right) + \ln d
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{b^2 - y}{1 + \frac{a}{r}} &= d \\
y = \left(u^1\right)^2 &= b^2 - d\left(1 + \frac{a}{r}\right)
\end{aligned}$$

Comparing to the previous result, $u^1 = \sqrt{b^2 - \left(1 + \frac{a}{r}\right)}$ we see that the results agree provided $d = 1$.

Solution to lowest order

To lowest order in $\frac{r_0}{r}$, the components of the velocity are

$$\begin{aligned}
u^0 &= \frac{b}{1 + \frac{a}{r}} \\
&= \left(1 + \frac{a}{2r_0}\right)\left(1 - \frac{a}{r}\right) \\
&= 1 + \frac{a}{2r_0} - \frac{a}{r} \\
&= 1
\end{aligned}$$

which gives $t = \tau$ and

$$u^1 = \sqrt{\frac{a}{r_0} - \frac{a}{r}}$$

Notice that the Newtonian acceleration in the r direction agrees with the lowest order limit if we fix the value of the constant a,

$$\begin{aligned}
-\frac{GM}{r^2 c^2} &= \frac{1}{c^2}\frac{d^2 r}{dt^2} \\
&= \frac{1}{c}\frac{du^1}{d\tau} \\
&= \frac{1}{2r^2}\frac{a}{1 + \frac{a}{r}}\left(b^2 - u^1 u^1\right) \\
&\approx \frac{a}{2r^2}\left(1 + \frac{a}{r_0} - \left(\frac{a}{r_0} - \frac{a}{r}\right)\right) \\
&\approx \frac{a}{2r^2} \\
\frac{a}{2r^2} &= -\frac{GM}{r^2 c^2} \\
a &= -\frac{2GM}{c^2}
\end{aligned}$$

This finally establishes the Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Orbits

The Kepler problem

For comparison, we first compute the orbits in Newtonian gravity. We start from the conservation laws. Since the velocity is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$$

the angular momentum is

$$\begin{aligned} \overrightarrow{L} &= \overrightarrow{r} \times \overrightarrow{p} \\ &= m\overrightarrow{r} \times \left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}\right) \\ &= mr\dot{\theta}\overrightarrow{r} \times \hat{\theta} + mr\sin\theta\dot{\varphi}\overrightarrow{r} \times \hat{\varphi} \end{aligned}$$

Since this is conserved in both magnitude and direction, the orbit remains in the plane perpendicular to $\overrightarrow{L}$. Without loss of generality, we may take the orbit to lie in the $\theta = \frac{\pi}{2}$ plane, so that

$$\begin{aligned} \overrightarrow{L} &= mr^2\dot{\varphi}\hat{k} \\ \vec{v} &= \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} \end{aligned}$$

The energy is also conserved,

$$\mathcal{E} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{GMm}{r}$$

Define the angular momentum per unit mass, $l = r^2\dot{\varphi}$ and the energy per unit mass, $E = \frac{\mathcal{E}}{m}$. Then we have $\dot{\varphi} = \frac{l}{r^2}$ so that

$$\begin{aligned} E &= \frac{1}{2}\left(\dot{r}^2 + \frac{l^2}{r^2}\right) - \frac{GM}{r} \\ \dot{r} &= \sqrt{2E - \frac{l^2}{r^2} + \frac{2GM}{r}} \end{aligned}$$

To find an equation for the orbit, $r(\varphi)$, divide by $\dot{\varphi} = \frac{l}{r^2}$ and integrate:

$$\varphi = \int \frac{ldr}{r^2\sqrt{2E - \frac{l^2}{r^2} + \frac{2GM}{r}}}$$

Now set $u = \frac{1}{r}$ so that

$$\begin{aligned} \varphi &= \int \frac{-ldu}{\sqrt{2E - l^2u^2 + 2GMu}} \\ &= \int \frac{-ldu}{\sqrt{-\left(\frac{GM}{l} - lu\right)^2 + 2E + \frac{G^2M^2}{l^2}}} \end{aligned}$$

Let

$$\begin{aligned} y &= \frac{GM}{l} - lu \\ A^2 &= 2E + \frac{G^2M^2}{l^2} \end{aligned}$$

so that

$$\varphi = \int \frac{dy}{\sqrt{-y^2 + A^2}}$$

so with $y = A\sin\theta$ we have

$$\begin{aligned} \varphi &= \int \frac{A\cos\theta d\theta}{A\cos\theta} \\ &= \theta \\ &= \arcsin\left(\frac{y}{A}\right) \end{aligned}$$

Solving for r we have,

$$\begin{aligned} A\sin\varphi &= y \\ &= \frac{GM}{l} - lu \\ &= \frac{GM}{l} - \frac{l}{r} \end{aligned}$$

so

$$\begin{aligned} \frac{1}{r} &= \frac{GM}{l^2} - \frac{A}{l}\sin\varphi \\ r &= \frac{\frac{l^2}{GM}}{1 - \sqrt{1 + \frac{2l^2E}{G^2M^2}}\sin\varphi} \end{aligned}$$

To see that this describes an ellipse, let

$$r = \frac{a}{1 - e\sin\varphi}$$

Then, changing to Cartesian coordinates,

$$\begin{aligned} r - er\sin\varphi &= a \\ r - ey &= a \\ r^2 &= a^2 + 2eay + e^2y^2 \\ x^2 + y^2 &= a^2 + 2eay + e^2y^2 \\ x^2 + y^2 - 2eay - ey^2 &= a^2 \\ x^2 + (1-e)\left(y - \frac{ea}{1-e}\right)^2 &= a^2\left(1 + \frac{e^2}{1-e}\right) \end{aligned}$$

Finally, setting $y_0 = \frac{ea}{1-e}$, $b = a^2\left(1 + \frac{e^2}{1-e}\right)$ and $c^2 = \frac{b^2}{1-e}$ we have the standard form for an ellipse centered at $(x, y) = (0, y_0)$:

$$\frac{x^2}{b^2} + \frac{(y - y_0)^2}{c^2} = 1$$

An examination of the magnitudes of the constants shows that this solution is valid for bound states, with $E < 0$. For positive energy, the final integral gives a hyperbolic function and the equation describes a hyperbola.

Now consider the orbits described by general relativity.

Geodesic orbits

The geodesic equations are:

$$
\begin{aligned}
0 &= \frac{1}{c}\frac{du^0}{d\tau} - \frac{a}{r^2\left(1+\frac{a}{r}\right)}u^0u^1 \\
0 &= \frac{1}{c}\frac{du^1}{d\tau} - \left(1+\frac{a}{r}\right)\frac{a}{2r^2}u^0u^0 + \frac{a}{2r^2\left(1+\frac{a}{r}\right)}u^1u^1 - r\left(1+\frac{a}{r}\right)u^2u^2 - \left(1+\frac{a}{r}\right)r\sin^2\theta u^3u^3 \\
0 &= \frac{1}{c}\frac{du^2}{d\tau} - \sin\theta\cos\theta u^3u^3 + \frac{2}{r}u^2u^1 \\
0 &= \frac{1}{c}\frac{du^3}{d\tau} + \frac{2\cos\theta}{\sin\theta}u^2u^3 + \frac{2}{r}u^3u^1
\end{aligned}
$$

We also have the relation given by the line element,

$$
\frac{ds^2}{d\tau^2} = -\left(1+\frac{a}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2\left(1+\frac{a}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{c^2}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{r^2}{c^2}\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2
$$

Because of the spherical symmetry, orbits will remain in a plane, as seen by choosing initial conditions with $\theta = \frac{\pi}{2}$ and $u^2(0) = 0$. Then the geodesic equations become

$$
\begin{aligned}
0 &= \frac{1}{c}\frac{du^0}{d\tau} - \frac{a}{r^2\left(1+\frac{a}{r}\right)}u^0u^1 \\
0 &= \frac{1}{c}\frac{du^1}{d\tau} - \left(1+\frac{a}{r}\right)\frac{a}{2r^2}u^0u^0 + \frac{a}{2r^2\left(1+\frac{a}{r}\right)}u^1u^1 - \left(1+\frac{a}{r}\right)ru^3u^3 \\
0 &= \frac{1}{c}\frac{du^2}{d\tau} \\
0 &= \frac{1}{c}\frac{du^3}{d\tau} + \frac{2}{r}u^3u^1
\end{aligned}
$$

so that u^2 remains zero. We may integrate the φ equation immediately,

$$
\begin{aligned}
\frac{du^3}{u^3} &= -\frac{2}{r}dr \\
u^3 &= \frac{l_0}{r^2} \\
r^2\frac{d\varphi}{d\tau} &= l_0
\end{aligned}
$$

which simply states conservation of angular momentum.

We also have the same result for u^0 as we did for radial geodesics,

$$
\left(1+\frac{a}{r}\right)u^0 = b
$$

though b differs. How we proceed depends on the type of orbit we desire.

Timelike orbits and perihelion advance

We can compute u^1 from the line element for a timelike orbit, $\frac{ds^2}{d\tau^2} = -1$,

$$
-1 = -\frac{b^2}{1+\frac{a}{r}} + \frac{1}{c^2\left(1+\frac{a}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + \frac{l_0^2}{r^2c^2}
$$

$$
\begin{aligned}
-1-\frac{a}{r} &= -b^2+\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2+\frac{l_0^2}{r^2c^2}\left(1+\frac{a}{r}\right) \\
\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 &= b^2-1-\frac{a}{r}-\frac{l_0^2}{r^2c^2}\left(1+\frac{a}{r}\right)
\end{aligned}
$$

To find b let $t=\tau=0$ when $\frac{dr}{d\tau}=0$. Then

$$
\begin{aligned}
-1 &= -\frac{b^2}{1+\frac{a}{r_0}}+\frac{l_0^2}{r_0^2c^2} \\
b^2 &= \left(1+\frac{l_0^2}{r_0^2c^2}\right)\left(1-\frac{2GM}{r_0c^2}\right) \\
b &= \sqrt{1+\frac{l_0^2}{r_0^2c^2}-\frac{2GM}{r_0c^2}-\frac{2GMl_0^2}{r_0^3c^4}}
\end{aligned}
$$

so that with this value for b and $a=-\frac{2GM}{c^2}$ we have

$$
\left(\frac{dr}{d\tau}\right)^2=-\frac{2GM}{r_0}+\frac{2GM}{r}+\frac{l_0^2}{r_0^2}\left(1-\frac{2GM}{r_0c^2}\right)-\frac{l_0^2}{r^2}\left(1-\frac{2GM}{rc^2}\right)
$$

We find the equation for the orbit by dividing by $\left(u^3\right)^2=\frac{l_0^2}{r^4}$,

$$
\begin{aligned}
\left(\frac{dr}{d\varphi}\right)^2 &= r^4\left(-\frac{2GM}{r_0l_0^2}+\frac{2GM}{rl_0^2}+\frac{1}{r_0^2}\left(1-\frac{2GM}{r_0c^2}\right)-\frac{1}{r^2}\left(1-\frac{2GM}{rc^2}\right)\right) \\
d\varphi &= \frac{dr}{r^2\sqrt{-\frac{2GM}{r_0l_0^2}+\frac{2GM}{rl_0^2}+\frac{1}{r_0^2}\left(1-\frac{2GM}{r_0c^2}\right)-\frac{1}{r^2}\left(1-\frac{2GM}{rc^2}\right)}}
\end{aligned}
$$

In terms of $u=\frac{1}{r}$ this becomes

$$
\begin{aligned}
d\varphi &= \frac{-\frac{1}{u^2}u^2du}{\sqrt{-\frac{2GMu_0}{l_0^2}+\frac{2GM}{l_0^2}u+u_0^2\left(1-\frac{2GM}{c^2}u_0\right)-u^2\left(1-\frac{2GMu}{c^2}\right)}} \\
\left(\frac{du}{d\varphi}\right)^2 &= \left(u_0^2\left(1-\frac{2GM}{c^2}u_0\right)-\frac{2GMu_0}{l_0^2}\right)+\frac{2GM}{l_0^2}u-u^2+\frac{2GMu^3}{c^2}
\end{aligned}
$$

which is the usual equation for an ellipse except for the last term.

Make the definitions

$$
\begin{aligned}
u' &= \frac{du}{d\varphi} \\
a &= u_0^2\left(1-\frac{2GM}{c^2}u_0\right)-\frac{2GMu_0}{l_0^2} \\
b &= \frac{2GM}{l_0^2} \\
c &= \frac{2GM}{c^2}
\end{aligned}
$$

Then we have

$$
\left(u'\right)^2=a+bu-u^2+cu^3
$$

or

$$d\varphi = \frac{-du}{\sqrt{a + bu - u^2 + cu^3}}$$

This agrees with the Kepler result, except for the last, cubic term in the denominator, which is very small for ordinary stars or planets.

If we integrate this about half of one orbit without the cubic term, we just get π. Multiplying by 2 we get a complete circuit, that is,

$$2\pi = -2 \int_{u_-}^{u_+} \frac{du}{\sqrt{a + bu - u^2}}$$

where $u_\pm$ are the extremes of the orbit. The perihelion advance is the difference between this integral and 2π,

$$\Delta\varphi = 2\pi - 2 \int_{u_-}^{u_+} \frac{du}{\sqrt{a + bu - u^2 + cu^3}}$$

Unfortunately, estimating this straightforwardly involves an elliptic integral. I haven't found any way to do it simpler than Weinberg's calculation in Gravitation and Cosmology. I won't repeat his calculation here.

The result for the perihelion advance of Mercury is about 43 seconds of arc per century, in excellent agreement with the experimental result.

Null geodesics

Now return to the solution for the geodesic, but consider the null case, $\frac{ds^2}{d\tau^2} = 0$. Then we still have

$$\begin{aligned} \left(1 + \frac{a}{r}\right) u^0 &= b = \sqrt{1 - \frac{2m}{r_0}} \\ r^2 \frac{d\varphi}{d\tau} &= l_0 = r_0 c \end{aligned}$$

where r_0 is the radius of closest approach. But now we have

$$0 = -\left(1 + \frac{a}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2\left(1 + \frac{a}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + \frac{r^2}{c^2}\left(\frac{d\theta}{d\tau}\right)^2 + \frac{r^2}{c^2}\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

$$0 = -\frac{b^2}{1 + \frac{a}{r}} + \frac{1}{c^2\left(1 + \frac{a}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + \frac{l_0^2}{r^2 c^2}$$

Therefore, with $a = -\frac{2m}{r}$

$$\left(\frac{dr}{d\tau}\right)^2 = b^2 c^2 - \frac{c^2 r_0^2}{r^2}\left(1 - \frac{2m}{r}\right)$$

At the radius of closest approach, this becomes

$$\begin{aligned} 0 &= b^2 c^2 - \frac{l_0^2}{r_0^2}\left(1 - \frac{2m}{r_0}\right) \\ 0 &= b^2 - \left(1 - \frac{2m}{r_0}\right) \end{aligned}$$

The procedure is the same. We convert to an equation relating r and φ. Setting $A = \frac{bc}{l_0} = \frac{1}{r_0}\sqrt{1 - \frac{2m}{r_0}}$, this leads to

$$\begin{aligned}
\frac{\frac{dr}{d\tau}}{\frac{d\varphi}{d\tau}} &= \frac{r^2}{r_0 c}\sqrt{b^2c^2 - \frac{l_0^2}{r^2}\left(1 - \frac{2m}{r}\right)} \\
d\varphi &= \frac{dr}{\frac{r^2}{r_0 c}\sqrt{b^2c^2 - \frac{l_0^2}{r^2}\left(1 - \frac{2m}{r}\right)}} \\
&= \frac{dr}{r\sqrt{\frac{r^2}{r_0^2}b^2 - \frac{l_0^2}{r_0^2c^2}\left(1 - \frac{2m}{r}\right)}} \\
&= \frac{dr}{r\sqrt{\frac{r^2}{r_0^2} - 1 + \frac{2m}{r} - \frac{r^2}{r_0^2}\frac{2m}{r_0}}} \\
&= \frac{dr}{r\sqrt{\frac{r^2}{r_0^2} - 1}\sqrt{1 + \frac{2m}{r_0}\frac{\frac{r_0^3}{r^3} - 1}{1 - \frac{r_0^2}{r^2}}}}
\end{aligned}$$

The total change in angle twice the integral from infinity to r_0. No deviation corresponds to a change in φ of π, so

$$\Delta\varphi = -\pi + 2\int \frac{dr}{r\sqrt{\frac{r^2}{r_0^2} - 1}\sqrt{1 + \frac{2m}{r_0}\frac{\frac{r_0^3}{r^3} - 1}{1 - \frac{r_0^2}{r^2}}}}$$

Now expand the second square root in powers of $\frac{m}{r_0} \ll 1$ and substitute $\frac{r_0}{r} = \sin\theta$,

$$\begin{aligned}
\Delta\varphi &= -\pi + 2\int \frac{dr}{r\sqrt{\frac{r^2}{r_0^2} - 1}}\left(1 + \frac{m}{r_0}\left(1 - \frac{r_0^3}{r^3}\right)\left(1 - \frac{r_0^2}{r^2}\right)^{-1} + \mathcal{O}\left(\frac{m^2}{r_0^2}\right)\right) \\
&= -\pi - 2\int \frac{r_0 dr}{r^2\sqrt{1 - \frac{r_0^2}{r^2}}}\left(1 + \frac{m}{r_0}\left(1 - \frac{r_0^3}{r^3}\right)\left(1 - \frac{r_0^2}{r^2}\right)^{-1}\right) \\
&= -\pi + 2\int_0^{\frac{\pi}{2}} d\theta\left(1 + \frac{m}{r_0}\left(1 - \sin^3\theta\right)\left(1 - \sin^2\theta\right)^{-1}\right) \\
&= -\pi + 2\int_0^{\frac{\pi}{2}} d\theta\left(1 + \frac{m}{r_0}\left(1 + \sin\theta + \sin^2\theta\right)\left(1 + \sin\theta\right)^{-1}\right) \\
&= \frac{2m}{r_0}\left(1 + \sin\theta\right)^{-1}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}\cos\theta + (\cos\theta - 2)\sin\frac{\theta}{2}\right)\Big|_0^{\frac{\pi}{2}} \\
&= \frac{2m}{2r_0}\left(\sqrt{2}\right)\frac{1}{\sqrt{2}}(-2) - \frac{2m}{r_0} \\
&= -\frac{4m}{r_0}
\end{aligned}$$

The integral was easily handled by the Wolfram online integrator.

For the sun,

$$\begin{aligned}
\frac{4m}{r_0} &= \frac{4GM}{R} \\
&= \frac{4 \times 6.67 \times 10^{-11} \times 1.99 \times 10^{30}}{6.96 \times 10^8}
\end{aligned}$$

$$
\begin{aligned}
&= 8.476 \times 10^{-6} rad \times 57.2957 deg/rad \times \frac{1}{3600} sec/deg \\
&= 1.748 \;\; sec
\end{aligned}
$$

so light is deflected by 1.75 seconds of arc. This is confirmed by experiment.