Cosmology

December 9, 2010

1 Isotropic, homogeneous 3-spaces

The basic assumption for cosmology, based on considerable experimental evidence, is that there exist homogeneous, isotropic spatial hypersurfaces. The metric for spacetime must then take the form

$$ds^2 = -dt^2 + a(t)^2 h_{ij} dx^i dx^j$$

where h_{ij} is any homogeneous, isotropic 3-metric.

In order to be isotropic about a point $r = 0$, a spatial 3-metric must take the form of a 2-sphere, together with arbitrary radial part,

$$dl^2 = f(r)^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

and homogeneity then requires the corresponding curvature (or any curvature scalar) to be constant. If we compute the curvature in an orthonormal frame field, $\mathbf{e}^i$, then all components of the curvature must be equal. An orthonormal frame is given by

$$\begin{aligned}
\mathbf{e}^r &= f\mathbf{d}r \\
\mathbf{e}^\theta &= r\mathbf{d}\theta \\
\mathbf{e}^\varphi &= r\sin\theta\mathbf{d}\varphi
\end{aligned}$$

The connection and curvature are given by

$$\begin{aligned}
\mathbf{de}^i &= \mathbf{e}^j \wedge \boldsymbol{\omega}^i{}_j \\
\mathbf{R}^i{}_j &= \mathbf{d}\boldsymbol{\omega}^i{}_j - \boldsymbol{\omega}^k{}_j \wedge \boldsymbol{\omega}^i{}_k
\end{aligned}$$

To solve for the spin connection, $\boldsymbol{\omega}^i{}_j$, we have the three equations

$$\begin{aligned}
\mathbf{de}^r &= \mathbf{e}^\theta \wedge \boldsymbol{\omega}^r{}_\theta + \mathbf{e}^\varphi \wedge \boldsymbol{\omega}^r{}_\varphi \\
\mathbf{de}^\theta &= \mathbf{e}^r \wedge \boldsymbol{\omega}^\theta{}_r + \mathbf{e}^\varphi \wedge \boldsymbol{\omega}^\theta{}_\varphi \\
\mathbf{de}^\varphi &= \mathbf{e}^r \wedge \boldsymbol{\omega}^\varphi{}_r + \mathbf{e}^\theta \wedge \boldsymbol{\omega}^\varphi{}_\theta
\end{aligned}$$

Substituting for the basis forms, these become

$$\begin{aligned}
0 &= r\mathbf{d}\theta \wedge \boldsymbol{\omega}^r{}_\theta + \mathbf{e}^\varphi \wedge \boldsymbol{\omega}^r{}_\varphi \\
\mathbf{d}r \wedge \mathbf{d}\theta &= f\mathbf{d}r \wedge \boldsymbol{\omega}^\theta{}_r + \mathbf{e}^\varphi \wedge \boldsymbol{\omega}^\theta{}_\varphi \\
\sin\theta\mathbf{d}r \wedge \mathbf{d}\varphi + r\cos\theta\mathbf{d}\theta \wedge \mathbf{d}\varphi &= f\mathbf{d}r \wedge \boldsymbol{\omega}^\varphi{}_r + r\mathbf{d}\theta \wedge \boldsymbol{\omega}^\varphi{}_\theta
\end{aligned}$$

Matching terms we see that we can solve the second by setting

$$\boldsymbol{\omega}^\theta{}_r = \frac{1}{f}\mathbf{d}\theta$$

and the third with

$$\begin{aligned}\boldsymbol{\omega}^{\varphi}{}_{r} &= \frac{1}{f}\sin\theta\mathbf{d}\varphi \\ \boldsymbol{\omega}^{\varphi}{}_{\theta} &= \cos\theta\mathbf{d}\varphi\end{aligned}$$

and using the antisymmetry $\boldsymbol{\omega}^{\theta}{}_{\varphi} = -\boldsymbol{\omega}^{\varphi}{}_{\theta}$, $\boldsymbol{\omega}^{r}{}_{\theta} = -\boldsymbol{\omega}^{\theta}{}_{r}$, and $\boldsymbol{\omega}^{r}{}_{\varphi} = -\boldsymbol{\omega}^{\varphi}{}_{r}$, we see that all three equations are satisfied.

Now compute the curvature 2-forms,

$$\mathbf{R}^{i}{}_{j} = \mathbf{d}\boldsymbol{\omega}^{i}{}_{j} - \boldsymbol{\omega}^{k}{}_{j} \wedge \boldsymbol{\omega}^{i}{}_{k}$$

We have

$$\begin{aligned}\mathbf{R}^{r}{}_{\theta} &= \mathbf{d}\boldsymbol{\omega}^{r}{}_{\theta} - \boldsymbol{\omega}^{k}{}_{\theta} \wedge \boldsymbol{\omega}^{r}{}_{k} \\ &= \mathbf{d}\left(-\frac{1}{f}\mathbf{d}\theta\right) - \boldsymbol{\omega}^{\varphi}{}_{\theta} \wedge \boldsymbol{\omega}^{r}{}_{\varphi} \\ &= \frac{f'}{f^2}\mathbf{d}r \wedge \mathbf{d}\theta - (\cos\theta\mathbf{d}\varphi) \wedge \left(-\frac{1}{f}\sin\theta\mathbf{d}\varphi\right) \\ &= \frac{f'}{rf^3}\mathbf{e}^{r} \wedge \mathbf{e}^{\theta}\end{aligned}$$

and

$$\begin{aligned}\mathbf{R}^{r}{}_{\varphi} &= \mathbf{d}\boldsymbol{\omega}^{r}{}_{\varphi} - \boldsymbol{\omega}^{k}{}_{\varphi} \wedge \boldsymbol{\omega}^{r}{}_{k} \\ &= \mathbf{d}\left(-\frac{1}{f}\sin\theta\mathbf{d}\varphi\right) - \boldsymbol{\omega}^{\theta}{}_{\varphi} \wedge \boldsymbol{\omega}^{r}{}_{\theta} \\ &= \frac{f'}{f^2}\sin\theta\mathbf{d}r \wedge \mathbf{d}\varphi - \frac{1}{f}\cos\theta\mathbf{d}\theta \wedge \mathbf{d}\varphi - (-\cos\theta\mathbf{d}\varphi) \wedge \left(-\frac{1}{f}\mathbf{d}\theta\right) \\ &= \frac{f'}{f^2}\sin\theta\mathbf{d}r \wedge \mathbf{d}\varphi - \frac{1}{f}\cos\theta\mathbf{d}\theta \wedge \mathbf{d}\varphi - \frac{1}{f}\cos\theta\mathbf{d}\varphi \wedge \mathbf{d}\theta \\ &= \frac{f'}{rf^3}\mathbf{e}^{r} \wedge \mathbf{e}^{\varphi}\end{aligned}$$

and finally

$$\begin{aligned}\mathbf{R}^{\theta}{}_{\varphi} &= \mathbf{d}\boldsymbol{\omega}^{\theta}{}_{\varphi} - \boldsymbol{\omega}^{k}{}_{\varphi} \wedge \boldsymbol{\omega}^{\theta}{}_{k} \\ &= \mathbf{d}(-\cos\theta\mathbf{d}\varphi) - \boldsymbol{\omega}^{r}{}_{\varphi} \wedge \boldsymbol{\omega}^{\theta}{}_{r} \\ &= \sin\theta\mathbf{d}\theta \wedge \mathbf{d}\varphi - \left(-\frac{1}{f}\sin\theta\mathbf{d}\varphi\right) \wedge \left(\frac{1}{f}\mathbf{d}\theta\right) \\ &= \left(1 - \frac{1}{f^2}\right)\sin\theta\mathbf{d}\theta \wedge \mathbf{d}\varphi \\ &= \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)\mathbf{e}^{\theta} \wedge \mathbf{e}^{\varphi}\end{aligned}$$

so the curvature components are

$$\begin{aligned}R^{r}{}_{\theta r\theta} &= \frac{f'}{rf^3} \\ R^{r}{}_{\varphi r\varphi} &= \frac{f'}{rf^3} \\ R^{\theta}{}_{\varphi\theta\varphi} &= \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)\end{aligned}$$

Since we are in an orthonormal frame, these must all be equal. Therefore,

$$\begin{aligned}
\frac{f'}{rf^3} &= \frac{1}{r^2}\left(1-\frac{1}{f^2}\right) \\
\int_{f_0}^{f} \frac{df}{f\,(f^2-1)} &= \int_{r_0}^{r} \frac{1}{r} dr \\
\int_{f_0}^{f} df \left(-\frac{1}{f} + \frac{1}{2\,(f-1)} + \frac{1}{2\,(f+1)}\right) &= \ln \frac{r}{r_0} \\
-\ln \frac{f}{f_0} + \frac{1}{2}\ln \frac{f^2-1}{f_0^2-1} &= \ln \frac{r}{r_0} \\
\frac{f_0^2}{f^2}\frac{f^2-1}{f_0^2-1} &= \frac{r^2}{r_0^2} \\
f^2\left(1-\frac{f_0^2-1}{r_0^2 f_0^2} r^2\right) &= 1
\end{aligned}$$

Let $\kappa = \frac{f_0^2-1}{r_0^2 f_0^2}$, noticing that κ may be any real number, positive or negative. Then the only homogeneous, isotropic 3-metrics may be put in the form

$$ds^2 = \frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Rescaling s and r by an arbitrary factor, the only effect is to change the value of κ by the square of that factor. This means that we can scale the r coordinate so that $\kappa = \pm 1$ or $\kappa = 0$, with the zero case giving flat space. For $\kappa = \pm 1$, the curvature scalar is full curvature tensor is proportional to

$$\begin{aligned}
\frac{f'}{rf^3} = \frac{1}{r^2}\left(1-\frac{1}{f}\right) &= \frac{1}{r^2}\left(1-\left(1-\kappa r^2\right)\right) \\
&= \kappa
\end{aligned}$$

Indeed, the entire curvature tensor may be written in terms of this single constant as

$$R^{ij}{}_{kl} = \kappa\left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right)$$

or, lowering an index,

$$R^i_{jkl} = \kappa\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right)$$

Notice that we have spaces of both constant negative curvature and constant positive curvature.

Finally, note that the change of coordinate,

$$r = \sin\chi$$

for $\kappa = 1$ or

$$r = \sinh\chi$$

for $\kappa = -1$ puts the line element in the more obviously hyperspherical form

$$ds^2 = d\chi^2 + \sin\chi^2 d\theta^2 + \sin\chi^2 \sin^2\theta d\varphi^2$$

or negative curvature form

$$ds^2 = d\chi^2 + \sinh\chi^2 d\theta^2 + \sinh\chi^2 \sin^2\theta d\varphi^2$$

2 Spacetime cosmological curvature

We may now write the metric for a cosmological model as

$$\begin{aligned} ds^2 &= -dt^2 + a^2\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right) \\ &= -dt^2 + a^2 h_{ij} dx^i dx^j \end{aligned}$$

where $\kappa = 0, \pm 1$, $i, j = 1, 2, 3$, and $a = a(t)$ sets the cosmic distance scale at any given time.

The connection is easily found,

$$\begin{aligned} -\Gamma_{0ij} = \Gamma_{i0j} = \Gamma_{ij0} &= a\dot{a}h_{ij} \\ \Gamma_{ijk} &= a^2\tilde{\Gamma}_{ijk} \end{aligned}$$

where $\tilde{\Gamma}_{ijk}$ is the connection found above for the maximally symmetric metric, h_{ij}. Therefore,

$$\begin{aligned} \Gamma^i_{0j} = \Gamma^i_{j0} &= \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^0_{ij} &= a\dot{a}h_{ij} \\ \Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} \end{aligned}$$

The curvature can also be written in terms of maximally symmetric parts, and parts depending on $a(t)$.

$$\begin{aligned} R^i_{jkl} &= \Gamma^i_{jl,k} - \Gamma^i_{jk,l} - \Gamma^i_{bl}\Gamma^b_{jk} + \Gamma^i_{bk}\Gamma^b_{jl} \\ &= \tilde{R}^i_{jkl} - \Gamma^i_{0l}\Gamma^0_{jk} + \Gamma^i_{0k}\Gamma^0_{jl} \\ &= \tilde{R}^i_{jkl} + \dot{a}^2\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right) \end{aligned}$$

and replacing $\tilde{R}^i_{jkl}$ with the expression above for the maximally symmetric curvature,

$$R^i_{jkl} = \left(\kappa + \dot{a}^2\right)\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right)$$

Next, consider

$$R^0_{jkl} = \Gamma^0_{jl,k} - \Gamma^0_{jk,l} - \Gamma^0_{bl}\Gamma^b_{jk} + \Gamma^0_{bk}\Gamma^b_{jl}$$

This must be proportional to some rank-3 tensor in the maximally symmetric space, but there is none so we expect these components to vanish. Indeed, we find

$$\begin{aligned} R^0_{jkl} &= a\dot{a}h_{jl,k} - a\dot{a}h_{jk,l} - a\dot{a}h_{ml}\tilde{\Gamma}^m_{jk} + a\dot{a}h_{ml}\tilde{\Gamma}^m_{jk} \\ &= a\dot{a}\left(h_{jl,k} - h_{ml}\tilde{\Gamma}^m_{jk} - h_{jm}\tilde{\Gamma}^m_{lk} - h_{jk,l} + h_{ml}\tilde{\Gamma}^m_{jk} + h_{jm}\tilde{\Gamma}^m_{kl}\right) \\ &\quad + h_{jm}\tilde{\Gamma}^m_{lk} - h_{jm}\tilde{\Gamma}^m_{kl} \\ &= a\dot{a}\left(h_{jl;k} - h_{jk;l}\right) \\ &= 0 \end{aligned}$$

where the derivatives of h_{ij} in the penultimate step are with respect to the maximally symmetric connection. Since h_{ij} is the metric compatible with this connection, the derivatives vanish.

The final components are

$$\begin{aligned} R^0_{j0l} &= \Gamma^0_{jl,0} - \Gamma^0_{j0,l} - \Gamma^0_{bl}\Gamma^b_{j0} + \Gamma^0_{b0}\Gamma^b_{jl} \\ &= \Gamma^0_{jl,0} - \Gamma^0_{bl}\Gamma^b_{j0} \\ &= \left(a\ddot{a} + \dot{a}^2\right)h_{jl} - a\dot{a}h_{ml}\frac{\dot{a}}{a}\delta^m_j \\ &= a\ddot{a}h_{jl} \end{aligned}$$

Collecting terms, we have

$$\begin{aligned} R^0_{j0l} &= a\ddot{a}h_{jl} \\ R^i_{jkl} &= \left(\kappa + \dot{a}^2\right)\left(\delta^i_k h_{jl} - \delta^i_l h_{jk}\right) \end{aligned}$$

and terms related to these by symmetry.

The Ricci tensor follows immediately,

$$\begin{aligned} R_{00} &= R^i_{0i0} \\ &= -\frac{1}{a^2}h^{ij}R^0_{j0i} \\ &= -\frac{1}{a^2}h^{ij}a\ddot{a}h_{ji} \\ &= -\frac{3\ddot{a}}{a} \end{aligned}$$

and

$$\begin{aligned} R_{ij} &= R^0_{i0j} + R^m_{imj} \\ &= a\ddot{a}h_{ij} + \left(\kappa + \dot{a}^2\right)\left(\delta^m_m h_{ij} - \delta^m_i h_{mj}\right) \\ &= \left(a\ddot{a} + 2\kappa + 2\dot{a}^2\right)h_{ij} \end{aligned}$$

and the Ricci scalar is

$$\begin{aligned} R &= g^{00}R_{00} + \frac{1}{a^2}h^{ij}R_{ij} \\ &= \frac{3\ddot{a}}{a} + \frac{3}{a^2}\left(a\ddot{a} + 2\kappa + 2\dot{a}^2\right) \\ &= \frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right) \end{aligned}$$

Finally, we have the components of the Einstein tensor,

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}g_{00}R \\ &= -\frac{3\ddot{a}}{a} + \frac{3\ddot{a}}{a} + \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) \\ &= \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) \end{aligned}$$

and

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\ &= \left(a\ddot{a} + 2\kappa + 2\dot{a}^2\right)h_{ij} - \frac{1}{2}a^2h_{ij}\left(\frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right)\right) \\ &= \left(a\ddot{a} + 2\kappa + 2\dot{a}^2 - 3a\ddot{a} - 3\left(\kappa + \dot{a}^2\right)\right)h_{ij} \\ &= -\left(2a\ddot{a} + \kappa + \dot{a}^2\right)h_{ij} \end{aligned}$$

3 The stress-energy tensor

We consider the stress-energy tensor of a perfect, isotropic fluid,

$$T^{ab} = (\rho + p)\,u^a u^b + pg^{ab}$$

Here the pressure must be the same in all directions. The 4-velocity $u^a = (1,0,0,0)$ is tangent to the co-moving geodesics, $u^a{}_{;b} u^b = 0$.

The conservation equation now gives

$$\begin{aligned} 0 &= T^{ab}{}_{;b} \\ &= (\rho + p)_{,b} u^a u^b + (\rho + p)\left(u^a{}_{;b} u^b + u^a u^b{}_{;b}\right) + p_{;b} g^{ab} \\ &= u^a \frac{d}{d\tau}(\rho + p) + (\rho + p) u^a u^b{}_{;b} + g^{ab} \frac{\partial p}{\partial x^b} \end{aligned}$$

Since ρ and p only depend on t, this reduces to

$$\begin{aligned} 0 &= u^0 (\dot{\rho} + \dot{p}) + (\rho + p) u^0 u^b{}_{;b} + g^{00} \frac{\partial p}{\partial t} \\ &= \dot{\rho} + \dot{p} + (\rho + p) u^b{}_{;b} - \dot{p} \end{aligned}$$

where, setting $h = \det h_{ij}$, the divergence of u^a is given by

$$\begin{aligned} u^b{}_{;b} &= \frac{1}{\sqrt{-g}} \partial_b \left(\sqrt{-g} u^b\right) \\ &= \frac{1}{a^3 h} \partial_b \left(h a^3 u^b\right) \\ &= \frac{1}{a^3} \partial_0 \left(a^3 u^0\right) \\ &= \frac{3\dot{a}}{a} \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &= \frac{d\rho}{d\tau} + (\rho + p) u^b{}_{;b} \\ &= \dot{\rho} + (\rho + p)\left(\frac{3\dot{a}}{a}\right) \end{aligned}$$

To complete the condition, we require an equation of state relating ρ and p. There are two relevant cases. At the present era, with the universe filled with chunks of massive stuff, the pressure is negligible and we may set $p = 0$. Then integrating,

$$\rho a^3 = const.$$

which simply says that in a volume $a^3(t)$ there is a constant amount of mass. For simplicity, define the constant to be m, so we have

$$\begin{aligned} m &\equiv \rho a^3 \\ T^{00} &\equiv \rho \\ &= \frac{m}{a^3(t)} \end{aligned}$$

The second relevant equation of state is that of radiation. This applies in the early universe when radiation fields were substantial and particle energies were so high that the energy and momentum are approximately equal, $E = \sqrt{\mathbf{p}^2 + m^2} \approx |\mathbf{p}|$. In this case

$$\rho = \frac{1}{3} p$$

and integration leads to

$$\rho a^4 = const.$$

In this case (even though ρ is clearly no longer a mass density), we still call the constant m, and we have

$$T^{00} = \frac{m}{a^4}$$

The Einstein equation

We may now write the Einstein equation, including a possible cosmological constant, Λ,

$$G_{ab} + \Lambda g_{ab} = \beta T_{ab}$$

where $\beta = \frac{8\pi G}{c^4}$. Then for the 00 component,

$$\begin{aligned} G_{00} + \Lambda g_{00} &= \beta T_{00} \\ \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) - \Lambda &= \frac{\beta m}{a^3} \end{aligned}$$

while the spatial components give a second equation,

$$\begin{aligned} G_{ij} + \Lambda g_{ij} &= 0 \\ -\left(2a\ddot{a} + \kappa + \dot{a}^2 - \Lambda a^2\right) h_{ij} &= 0 \end{aligned}$$

These two equations are not independent, but are related by the Bianchi identity. To see this, multiply the 00 equation by a^2 and differentiate with respect to time,

$$\begin{aligned} 3\left(\kappa + \dot{a}^2\right) - \Lambda a^2 &= \frac{\beta m}{a} \\ 6\dot{a}\ddot{a} - 2\Lambda a\dot{a} &= -\frac{\beta m}{a}\frac{\dot{a}}{a} \end{aligned}$$

Substituting the original equation for $\frac{\beta m}{a}$ and simplifying,

$$\begin{aligned} 6\dot{a}\ddot{a} - 2\Lambda a\dot{a} &= -\frac{\dot{a}}{a}\left(3\left(\kappa + \dot{a}^2\right) - \Lambda a^2\right) \\ 6\ddot{a} - 2\Lambda a &= -\frac{3}{a}\left(\kappa + \dot{a}^2\right) + \Lambda a \end{aligned}$$

and therefore, multiplying by $\frac{a}{3}$,

$$2a\ddot{a} - \Lambda a^2 + \kappa + \dot{a}^2 = 0$$

reproducing the spatial equation.

We have therefore reduced this cosmological model to the conservation law together with a single equation, called the Friedmann equation,

$$3a\left(\kappa + \dot{a}^2\right) - \Lambda a^3 - \beta m = 0$$

Curvature singularity

The metric appears to be degenerate if $a = 0$ or if a diverges. If we look at the scalar curvature,

$$R = \frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right)$$

we see that the first of these, $a = 0$, is also a curvature singularity. The field equations,

$$\begin{aligned} 2a\ddot{a} + \kappa + \dot{a}^2 - \Lambda a^2 &= 0 \\ \frac{3}{a^2}\left(\kappa + \dot{a}^2\right) - \Lambda - \frac{\beta m}{a^3} &= 0 \end{aligned}$$

allow us to rewrite $\kappa + \dot{a}^2$ and the acceleration in terms of a,

$$\begin{aligned} \kappa + \dot{a}^2 &= \frac{a^2}{3}\left(\Lambda + \frac{\beta m}{a^3}\right) \\ \ddot{a} &= \frac{1}{2a}\left(\Lambda a^2 - \left(\kappa + \dot{a}^2\right)\right) \\ &= \frac{1}{3}\Lambda a - \frac{\beta m}{6a^2} \end{aligned}$$

so the scalar curvature becomes,

$$\begin{aligned} R &= \frac{6\ddot{a}}{a} + \frac{6}{a^2}\left(\kappa + \dot{a}^2\right) \\ &= \frac{6}{a}\left(\frac{1}{3}\Lambda a - \frac{\beta m}{6a^2}\right) + 2\left(\Lambda + \frac{\beta m}{a^3}\right) \\ &= 4\Lambda + \frac{\beta m}{a^3} \end{aligned}$$

which diverges if and only if $a = 0$.

It is not hard to show that other curvature invariants also diverge at $a = 0$, and nowhere else.

Properties of the Friedmann equation

Consider the Firedmann equation,

$$3a\kappa + 3a\dot{a}^2 - \Lambda a^3 - \beta m = 0$$

Even without integrating this, we can analyze the possible histories it describes. Dividing by $8\pi a^3$, we write it in terms of the Hubble parameter,

$$\begin{aligned} \frac{3\kappa}{8\pi a^2} + \frac{3\dot{a}^2}{8\pi a^2} - \frac{1}{8\pi}\Lambda - \frac{m}{a^3} &= 0 \\ \frac{3}{8\pi a^2}\kappa + \frac{3H^2}{8\pi} - \frac{1}{8\pi}\Lambda - \rho_m &= 0 \end{aligned}$$

Solve for the curvature constant term,

$$\frac{3}{8\pi a^2}\kappa = -\frac{3H^2}{8\pi} + \rho_m + \frac{1}{8\pi}\Lambda$$

and think of each term as a density,

$$\begin{aligned} \rho_\Lambda &= \frac{1}{8\pi}\Lambda \\ \rho_C &= \frac{3H^2}{8\pi} \end{aligned}$$

where ρ_C is called the critical density. Then

$$\frac{3}{8\pi a^2}\kappa = -\rho_C + \rho_m + \rho_\Lambda$$

If the current density of matter, including the effective matter density of the cosmological constant, $\rho_m + \rho_\Lambda$ exceeds the critical density, then the right side is positive and we must have $\kappa = 1$. The universe is then closed and we expect it to recollapse.

On the other hand, if $\rho_m + \rho_\Lambda$ is less than the critical density, κ must be negative and the spatial hypersurfaces are open.

As the universe expands, two of the terms become smaller,

$$\begin{aligned}\lim_{a\to\infty}\frac{3}{8\pi a^2}\kappa &= 0\\ \lim_{a\to\infty}\rho_m &= 0\end{aligned}$$

so in the late-time limit

$$\begin{aligned}H^2=\frac{\dot{a}^2}{a^2} &= \frac{1}{3}\Lambda\\ a(t) &= a_0e^{ut}\\ u &= \sqrt{\frac{\Lambda}{3}}\end{aligned}$$

The expansion becomes exponential with a rate governed by the cosmological constant. This is faster than the rate without cosmological constant. For example, with flat spacelike hypersurfaces, $\kappa=0$, and no cosmological constant, we have

$$\begin{aligned}\frac{3\dot{a}^2}{8\pi a^2} &= \frac{m}{a^3}\\ \dot{a} &= \sqrt{\frac{8\pi m}{3a}}\\ \sqrt{a}da &= \sqrt{\frac{8\pi m}{3}}(t-t_0)\\ a^{3/2}-a_0^{3/2} &= \sqrt{\frac{8\pi m}{3}}(t-t_0)\\ a &= \left(a_0^{3/2}+\sqrt{\frac{8\pi m}{3}}(t-t_0)\right)^{2/3}\end{aligned}$$

Asymptotically, this gives a simple power law,

$$a=\left(\frac{8\pi m}{3}\right)^{1/3}t^{2/3}$$

Since κ is, in fact, close to zero, while at the present time ρ_m is much larger than ρ_Λ, the rate of expansion is close to this power law. But as the universe continues to expand and ρ_m becomes negligible, the cosmological constant will lead to an exponential expansion, causing the expansion to speed up.

Properties of the Friedmann equation

We now examine the Friedmann equation,

$$3a\left(\kappa+\dot{a}^2\right)-\Lambda a^3-\beta m=0$$

Solving for $\dot{a}$,

$$\dot{a}=\pm\sqrt{\frac{\Lambda a^3+\beta m-3\kappa a}{3a}}$$

The rate of change of a is therefore divergent at $a=0$, and when a diverges.

We need to consider 4 cases, depending on the signs of Λ and κ. We may always take $a>0$.

Case 1: Positive spatial curvature and positive cosmological constant

When $\kappa = 1$ and $\Lambda > 0$,

$$\dot{a} = \pm\sqrt{\frac{1}{3}\Lambda a^2 + \frac{\beta m}{3a} - 1}$$

Consider the turning points,

$$\Lambda a^3 - 3a\kappa + \beta m = 0$$

This cubic equation has extrema when

$$\begin{aligned} \Lambda a^2 - 1 &= 0 \\ a &= +\frac{1}{\sqrt{\Lambda}} \end{aligned}$$

This only exists when the cosmological constant is nonzero and m and Λ are large enough to keep the argument of the radical positive. In this case, as a increases from zero, $\dot{a}$ decreases from infinity to a finite positive value, then increases again, becoming asymptotically linear in a,

$$\dot{a} \to \sqrt{\frac{\Lambda}{3}} a$$

This corresponds to a universe that expands from a big bang, slows its expansion as long as gravity dominates, then expands faster and faster in response to the cosmological constant at an exponential rate:

$$a = Ae^{\sqrt{\frac{\Lambda}{3}}t}$$

When the cosmological constant is too small to keep the argument positive, the universe expands from a big bang, slows to a stop, then recontracts to a big crunch.

Case 2: Positive spatial curvature and negative cosmological constant

When $\kappa = 1$ and $\Lambda < 0$,

$$\dot{a} = \pm\sqrt{\frac{\beta m}{3a} - 1 - \frac{1}{3}|\Lambda| a^2}$$

the universe expands from a big bang, but always reaches a point where $\dot{a} = 0$ and the contraction stops. The universe then recontracts to a big crunch after a finite time. It is easy to check that when $\dot{a} = 0$, the acceleration is negative,

$$\begin{aligned} \ddot{a} &= \frac{1}{2a}\left(\Lambda a^2 - \left(\kappa + \dot{a}^2\right)\right) \\ &= \frac{1}{2a}\left(-|\Lambda| a^2 - 1\right) \\ &< 0 \end{aligned}$$

so that if $\dot{a}$ is positive and slows to zero it will go negative and the universe will collapse. If $\dot{a}$ starts out negative, it will stay negative to the singularity.

Case 3: Negative spatial curvature and positive cosmological constant

When $\kappa = -1$ and the cosmological constant is positive, we have

$$\dot{a} = \pm\sqrt{\frac{\beta m}{3a} + 1 + \frac{1}{3}\Lambda a^2}$$

there argument of the radical is always positive, and $\dot{a} \neq 0$. Therefore, if $\dot{a} > 0$, the universe expands rapidly from a big bang, slowing until the cosmological constant leads to a late-time exponential, $a \to \exp\left(\sqrt{\frac{1}{3}\Lambda}\right) t$. If the universe begins with $\dot{a} < 0$ it continues contracting, following the time-reverse of the $\dot{a} > 0$ case.

Case 4: Negative spatial curvature and negative cosmological constant

When $\kappa = -1$ and the cosmological constant is positive, we have

$$\dot{a} = \pm\sqrt{\frac{\beta m}{3a} + 1 - \frac{1}{3}\left|\Lambda\right| a^2}$$

so that for small a the universe expands from a big bang. As it expands the matter contribution becomes less important and the cosmological constant drives $\dot{a}$ to zero. When this happens, the acceleration is given by

$$\begin{aligned}\ddot{a} &= \frac{1}{2a}\left(\Lambda a^2 - \left(\kappa + \dot{a}^2\right)\right) \\ &= \frac{1}{2a}\left(-\left|\Lambda\right| a^2 + 1\right)\end{aligned}$$

Since we have

$$0 = \frac{\beta m}{3a} + 1 - \frac{1}{3}\left|\Lambda\right| a^2$$

we can write this as

$$\begin{aligned}\ddot{a} &= \frac{1}{2a}\left(-3\left(\frac{\beta m}{3a} + 1\right) + 1\right) \\ &= \frac{1}{2a}\left(-\frac{\beta m}{a} - 2\right) \\ &< 0\end{aligned}$$

and the acceleration must change sign and recollapse.