Geometry of the 2-sphere

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1 The metric

The easiest way to find the metric of the 2-sphere (or the sphere in any dimension) is to picture it as embedded in one higher dimension of Euclidean space, then restrict to constant radius.

The 3-dim Euclidean metric in spherical coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

so restricting to

$$r = R = const.$$

gives

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\varphi^2$$

This is the metric we will study. As a matrix,

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

with inverse

$$g^{ij} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2\theta} \end{pmatrix}$$

There are more intrinsic ways to get this metric. One approach is to specify the symmetries we require – three independent rotations. There are techniques for finding the most general metric with given symmetry, so we can derive this form directly. Alternatively, we could ask for 2-dim spaces of constant curvature. Computing the metric for a general 2-geometry, then imposing constant curvature gives a set of differential equations that will lead to this form.

2 Curvature: a plan

One definition of curvature starts by carrying a vector by parallel transport around a closed loop. In general, the vector returns rotated from its original direction. The difference between this angle and the angle expected in a flat geometry is called the angular deficit. Next, calculate the area enclosed in the loop. Then the curvature at point Q is the limit of the angular deficit per unit area, as the loop shrinks to Q and the area to zero.

Explicitly, consider a closed curve $C : \lambda \in \mathcal{R} \rightarrow \mathcal{M}^2$, with tangent vector u^α at each point. Let v_0^i be the components of an arbitrary vector at a point $\mathcal{P}$ of the curve, and define the family of vectors $v^\alpha(\lambda)$ around the curve as the parallel transport of v_0^i along u^i:

$$\begin{aligned} 0 &= u^i D_i v^j \\ &= u^i \left(\partial_i v^j + v^k \Gamma^j_{ki} \right) \end{aligned}$$

Since we have the metric, we can compute Γ^j_{ki}, so as soon as we specify the curve, we can solve this equation for $v^i(\lambda)$. Then we can find the angle of rotation, α, by taking the inner product of $v^i(\lambda_{final})$ with $v^i(\lambda_{initial})$, where we have

$$\cos\alpha = \frac{g_{ij} v_0^i v^j (\lambda_{final})}{g_{ij} v_0^i v_0^j}$$

Then the angular deficit is

$$\Delta = 2\pi - \alpha$$

since transport around a closed loop in flat space will rotate the vector by 2π.

For the area inside the loop, we integrate the 2-dim volume element. This is given by the square root of the determinant of the metric, $\sqrt{g} = \sqrt{\det(g_{ij})}$, so that

$$\begin{aligned} A &= \int\int \sqrt{g}\, d\varphi\, d\theta \\ &= R^2 \int\int \sin\theta\, d\varphi\, d\theta \end{aligned}$$

3 Parallel transport on a non-geodesic circle

3.1 The curve

Now consider a circle around the sphere at constant θ_0 (e.g., constant latitude on the surface of Earth). We can parameterize the curve by the angle φ, so the curve is given by

$$x^i = (\theta_0, \varphi)$$

A vector tangent to the curve is

$$\begin{aligned} t^i &= \frac{dx^i}{d\varphi} \\ &= (0, 1) \end{aligned}$$

The length of this tangent vector is given by

$$\begin{aligned} l^2 &= g_{ij} t^i t^j \\ &= R^2 \sin^2\theta_0 \end{aligned}$$

so the unit tangent is

$$u^i = \frac{1}{R\sin\theta_0}\,(0,1)$$

3.2 The connection

We also need the connection. We have

$$\begin{aligned}
\Gamma^i_{jk} &= g^{im}\Gamma_{mjk} \\
\Gamma_{mjk} &= \frac{1}{2}\left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right)
\end{aligned}$$

Since the metric only has one non-constant component, $g_{\varphi\varphi}$, and that one depends only on θ, the only non-vanishing derivative of the metric is $g_{\varphi\varphi,\theta}$. This means that the only non-vanishing Γ^i_{jk} must have two φs and one θ index. Using the symmetry of the connection, we have

$$\begin{aligned}
\Gamma_{\varphi\varphi\theta} &= \Gamma_{\varphi\theta\varphi} = \frac{1}{2}\left(g_{\varphi\varphi,\theta} + g_{\varphi\theta,\varphi} - g_{\varphi\theta,\varphi}\right) \\
&= \frac{1}{2} g_{\varphi\varphi,\theta} \\
&= R^2 \sin\theta\cos\theta \\
\Gamma_{\theta\varphi\varphi} &= \frac{1}{2}\left(g_{\theta\varphi,\varphi} + g_{\theta\varphi,\varphi} - g_{\varphi\varphi,\theta}\right) \\
&= -R^2\sin\theta\cos\theta
\end{aligned}$$

Raising the first index is easy because the metric is diagonal. We have simply

$$\begin{aligned}
\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} &= g^{\varphi\varphi}\Gamma_{\varphi\varphi\theta} \\
&= \frac{1}{R^2\sin^2\theta} R^2 \sin\theta\cos\theta \\
&= \frac{\cos\theta}{\sin\theta} \\
\Gamma^\theta_{\varphi\varphi} &= g^{\theta\theta}\Gamma_{\theta\varphi\varphi} \\
&= -\frac{1}{R^2} R^2\sin\theta\cos\theta \\
&= -\sin\theta\cos\theta
\end{aligned}$$

3.3 Parallel transport

The parallel transport equation is

$$\begin{aligned}
0 &= u^i D_i v^j \\
&= u^i\left(\partial_i v^j + v^k \Gamma^j_{ki}\right) \\
&= \frac{1}{R\sin\theta_0}\left(\partial_\varphi v^j + v^k\Gamma^j_{k\varphi}\right)
\end{aligned}$$

There are two components to check. For $j = \theta$ we have

$$\begin{aligned}
0 &= \frac{1}{R\sin\theta_0}\left(\partial_\varphi v^\theta + v^k \Gamma^\theta_{k\varphi}\right) \\
&= \frac{1}{R\sin\theta_0}\left(\frac{\partial v^\theta}{\partial\varphi} + v^\varphi \Gamma^\theta_{\varphi\varphi}\right) \\
&= \frac{1}{R\sin\theta_0}\left(\frac{\partial v^\theta}{\partial\varphi} - v^\varphi \sin\theta_0\cos\theta_0\right)
\end{aligned}$$

For $j = \varphi$,

$$\begin{aligned}
0 &= \frac{1}{R\sin\theta_0}\left(\partial_\varphi v^\varphi + v^k \Gamma^\varphi_{k\varphi}\right) \\
&= \frac{1}{R\sin\theta_0}\left(\frac{\partial v^\varphi}{\partial\varphi} + v^\theta \frac{\cos\theta_0}{\sin\theta_0}\right)
\end{aligned}$$

Therefore, we need to solve the coupled equations,

$$\begin{aligned}
0 &= \frac{\partial v^\theta}{\partial\varphi} - v^\varphi \sin\theta_0\cos\theta_0 \\
0 &= \frac{\partial v^\varphi}{\partial\varphi} + v^\theta \frac{\cos\theta_0}{\sin\theta_0}
\end{aligned}$$

Taking a second derivative of the first equation and substituting the second,

$$\begin{aligned}
0 &= \frac{\partial^2 v^\theta}{\partial\varphi^2} - \frac{\partial v^\varphi}{\partial\varphi}\sin\theta_0\cos\theta_0 \\
&= \frac{\partial^2 v^\theta}{\partial\varphi^2} + v^\theta \frac{\cos\theta_0}{\sin\theta_0}\sin\theta_0\cos\theta_0 \\
&= \frac{\partial^2 v^\theta}{\partial\varphi^2} + v^\theta\cos^2\theta_0
\end{aligned}$$

Similarly, differentiating the second equation and substituting the first we have

$$\begin{aligned}
0 &= \frac{\partial^2 v^\varphi}{\partial\varphi^2} + \frac{\partial v^\theta}{\partial\varphi}\frac{\cos\theta_0}{\sin\theta_0} \\
&= \frac{\partial^2 v^\varphi}{\partial\varphi^2} + v^\varphi \sin\theta_0\cos\theta_0\frac{\cos\theta_0}{\sin\theta_0} \\
&= \frac{\partial^2 v^\varphi}{\partial\varphi^2} + v^\varphi\cos^2\theta_0
\end{aligned}$$

Each of these is just the equation for sinusoidal oscillation, so we may immediately write the solution,

$$\begin{aligned}
v^\theta(\varphi) &= A\cos\alpha\varphi + B\sin\alpha\varphi \\
v^\varphi(\varphi) &= C\cos\alpha\varphi + D\sin\alpha\varphi
\end{aligned}$$

where

$$\beta = \cos\theta_0$$

Starting the curve at $\varphi = 0$, it will close at $\varphi = 2\pi$. Then for v^α we have the initial condition $v^\alpha(0) = \left(v_0^\theta, v_0^\varphi\right)$, and from the original differential equations we must have

$$\begin{aligned}
\left.\frac{\partial v^\theta}{\partial\varphi}\right|_{\varphi=0} &= v_0^\varphi \sin\theta_0\cos\theta_0 \\
\left.\frac{\partial v^\varphi}{\partial\varphi}\right|_{\varphi=0} &= -v_0^\theta \frac{\cos\theta_0}{\sin\theta_0}
\end{aligned}$$

These conditions determine the constants A, B, C, D to be

$$\begin{aligned}
v^\theta(\varphi) &= v_0^\theta \cos\beta\varphi + \frac{v_0^\varphi \sin\theta_0\cos\theta_0}{\beta}\sin\beta\varphi \\
&= v_0^\theta \cos\beta\varphi + v_0^\varphi \sin\theta_0 \sin\beta\varphi \\
v^\varphi(\varphi) &= v_0^\varphi \cos\beta\varphi - v_0^\theta \frac{\sin\beta\varphi}{\sin\theta_0}
\end{aligned}$$

This gives the form of the transported vector at any point around the circle.

3.4 Norm of v

We have claimed that the norm of a vector is not changed by parallel transport. We can check this in the current example. The initial squared norm of $v^\alpha(0)$ is

$$|\vec{v}_0|^2 = R^2\left(v_0^\theta\right)^2 + R^2\sin^2\theta_0\left(v_0^\varphi\right)^2$$

while the norm of

$$\begin{aligned}
|\vec{v}|^2 &= R^2\left(v_0^\theta\cos\beta\varphi + v_0^\varphi\sin\theta_0\sin\beta\varphi\right)^2 + R^2\sin^2\theta_0\left(v_0^\varphi\cos\beta\varphi - v_0^\theta\frac{\sin\beta\varphi}{\sin\theta_0}\right)^2 \\
&= R^2\left(\left(v_0^\theta\right)^2\cos^2\beta\varphi + 2v_0^\theta v_0^\varphi\sin\theta_0\cos\beta\varphi\sin\beta\varphi + \left(v_0^\varphi\right)^2\sin^2\theta_0\sin^2\beta\varphi\right) \\
&\quad + R^2\sin^2\theta_0\left(\left(v_0^\varphi\right)^2\cos^2\beta\varphi - v_0^\theta v_0^\varphi\cos\beta\varphi\frac{\sin\beta\varphi}{\sin\theta_0} + \left(v_0^\theta\right)^2\frac{\sin^2\beta\varphi}{\sin^2\theta_0}\right) \\
&= R^2\left(v_0^\theta\right)^2\cos^2\beta\varphi + 2R^2v_0^\theta v_0^\varphi\sin\theta_0\cos\beta\varphi\sin\beta\varphi + R^2\left(v_0^\varphi\right)^2\sin^2\theta_0\sin^2\beta\varphi \\
&\quad + R^2\left(v_0^\varphi\right)^2\sin^2\theta_0\cos^2\beta\varphi - R^2v_0^\theta v_0^\varphi\sin\theta_0\cos\beta\varphi\sin\beta\varphi + R^2\left(v_0^\theta\right)^2\sin^2\beta\varphi \\
&= R^2\left(v_0^\theta\right)^2\left(\cos^2\beta\varphi + \sin^2\beta\varphi\right) + R^2\left(v_0^\varphi\right)^2\sin^2\theta_0\left(\sin^2\beta\varphi + \cos^2\beta\varphi\right) \\
&= R^2\left(v_0^\theta\right)^2 + R^2\left(v_0^\varphi\right)^2\sin^2\theta_0 \\
&= |\vec{v}_0|^2
\end{aligned}$$

which, as claimed, is independent of φ.

Now we turn to the calculation of the curvature.

3.5 Curvature of the 2-sphere

We need the angular deficit and the area of the sphere enclosed by the circular path.

3.5.1 The angular deficit

The angular deficit is given by

$$\Delta = 2\pi - \alpha$$

where the angle of rotatoin, α, is given by

$$\begin{aligned}\cos\alpha &= \frac{g_{ij}v_0^i v^j\left(\lambda_{final}\right)}{g_{ij}v_0^i v_0^j} \\ &= \frac{g_{ij}v_0^i v^j\left(2\pi\right)}{g_{ij}v_0^i v_0^j}\end{aligned}$$

The inner product in the numerator is

$$\begin{aligned}g_{ij}v_0^i v^j\left(2\pi\right) &= R^2\left(v_0^\theta v_{2\pi}^\theta + v_0^\varphi v_{2\pi}^\varphi \sin^2\theta_0\right) \\ &= R^2\left(\left(v_0^\theta v_0^\theta \cos\beta\varphi + v_0^\theta v_0^\varphi \sin\theta_0 \sin\beta\varphi\right) + v_0^\varphi\left(v_0^\varphi \cos\beta\varphi - v_0^\theta \frac{\sin\beta\varphi}{\sin\theta_0}\right)\sin^2\theta_0\right) \\ &= R^2\left(v_0^\theta v_0^\theta \cos\beta\varphi + v_0^\theta v_0^\varphi \sin\theta_0 \sin\beta\varphi + \left(v_0^\varphi\right)^2 \sin^2\theta_0 \cos\beta\varphi - v_0^\varphi v_0^\theta \sin\theta_0 \sin\beta\varphi\right) \\ &= R^2\left(\left(v_0^\theta\right)^2 + \left(v_0^\varphi\right)^2\sin^2\theta_0\right)\cos\beta\varphi \\ &= \left|\vec{v}_0\right|^2 \cos\beta\varphi\end{aligned}$$

The angle of rotation is therefore $\alpha = \beta = 2\pi\cos\theta_0$. Therefore, the angular deficit is

$$\begin{aligned}\Delta &= 2\pi - \alpha \\ &= 2\pi\left(1 - \cos\theta_0\right)\end{aligned}$$

3.5.2 The area enclosde by the loop

The area enclosed by the loop is

$$\begin{aligned}A &= \int d\varphi \int d\theta\sqrt{g} \\ &= R^2\int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin\theta d\theta \\ &= -2\pi R^2 \cos\theta|_0^{\theta_0} \\ &= -2\pi R^2\left(\cos\theta_0 - 1\right) \\ &= 2\pi R^2\left(1 - \cos\theta_0\right)\end{aligned}$$

3.5.3 The curvature

The curvature is now given by the limit as we shrink the loop to a point,

$$\begin{aligned} Curvature &= \lim_{\theta_0 \to 0} \frac{\Delta}{A} \\ &= \lim_{\theta_0 \to 0} \frac{2\pi (1 - \cos\theta_0)}{2\pi R^2 (1 - \cos\theta_0)} \\ &= \frac{1}{R^2} \end{aligned}$$

This increases as the sphere shrinks, which indeed makes the curvature greater.

4 Comparison with the Riemann curvature tensor

We can also compute the curvature using the Riemann curvature tensor. We already have the connection,

$$\begin{aligned} \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} &= \frac{\cos\theta}{\sin\theta} \\ \Gamma^{\theta}_{\varphi\varphi} &= -\sin\theta\cos\theta \end{aligned}$$

so it is straightforward to compute the curvature using

$$R^i_{jkm} = \Gamma^i_{jm,k} - \Gamma^i_{jk,m} + \Gamma^i_{nk}\Gamma^n_{jm} - \Gamma^i_{nm}\Gamma^n_{jk}$$

Since $R_{ijkm} = -R_{jikm} = -R_{ijmk}$, there is only one independent component. All of the rest follow from the symmetries of the curvature tensor. We can compute any one non-vanishing component. Choose

$$\begin{aligned} R^{\theta}_{\varphi\theta\varphi} &= \Gamma^{\theta}_{\varphi\varphi,\theta} - \Gamma^{\theta}_{\varphi\theta,\varphi} + \Gamma^{\theta}_{n\theta}\Gamma^{n}_{\varphi\varphi} - \Gamma^{\theta}_{n\varphi}\Gamma^{n}_{\varphi\theta} \\ &= (-\sin\theta\cos\theta)_{,\theta} - 0 + 0 \cdot \Gamma^{n}_{\varphi\varphi} - \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} \\ &= \left(-\cos^2\theta + \sin^2\theta\right) - (-\sin\theta\cos\theta)\left(\frac{\cos\theta}{\sin\theta}\right) \\ &= \sin^2\theta \end{aligned}$$

The full curvature tensor may be written in terms of the metric and Kronecker delta by including all the necessary symmetries,

$$R^i_{jkm} = \frac{1}{R^2}\left(\delta^i_k g_{jm} - \delta^i_m g_{jk}\right)$$

Then check that

$$\begin{aligned} R^{\theta}_{\varphi\theta\varphi} &= \frac{1}{R^2}\left(\delta^{\theta}_{\theta} g_{\varphi\varphi} - \delta^{\theta}_{\varphi} g_{\varphi\theta}\right) \\ &= \frac{1}{R^2} g_{\varphi\varphi} \\ &= \sin^2\theta \end{aligned}$$

Since the expression in terms of the metric has all the right symmetries, and the value of the one independent component is correct, it gives the full curvature tensor.

We may find the Ricci tensor by contraction:

$$
\begin{aligned}
R_{jm} &\equiv R^i_{jim} \\
&= \frac{1}{R^2}\left(\delta^i_i g_{jm} - \delta^i_m g_{ji}\right) \\
&= \frac{1}{R^2}\left(2g_{jm} - g_{jm}\right) \\
&= \frac{1}{R^2} g_{jm}
\end{aligned}
$$

The Ricci scalar is the contraction of this. Using the inverse metric,

$$
\begin{aligned}
R &\equiv g^{jm} R_{jm} \\
&= \frac{1}{R^2} g^{jm} g_{jm} \\
&= \frac{2}{R^2}
\end{aligned}
$$

which differs from our angular deficit formula only by an overall constant.