Practice with exterior calculus

1. Work out the form of the Laplacian of a 3-form, for arbitrary n.

2. Define the following differential forms to characterize the electric and magnetic fields:

$$\begin{aligned}
\boldsymbol{E} &= E^i g_{ij} \mathbf{d}x^j = E_j \mathbf{d}x^j \\
\boldsymbol{B} &= \frac{1}{2} B^i e_{ijk} \mathbf{d}x^j \wedge \mathbf{d}x^k \\
\boldsymbol{J} &= J^i g_{ij} \mathbf{d}x^j
\end{aligned}$$

   where E^i, B^i, and J^i are the covariant components in a vector basis. Rewrite Maxwell's equations,

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{4\pi}{c}\rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}
\end{aligned}$$

   in terms of $\mathbf{E}, \mathbf{B}, \mathbf{J}$ and a function, ρ.

3. Write and prove expressions in terms of differential forms which are equivalent to each of the following (from J. D. Jackson, *Classical Electrodynamics*). Here $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are arbitrary vectors, and ψ is an arbitrary function. You don't need to translate between vector and form expressions – just write the equivalent expression using forms and prove it.

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} \\
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \\
\nabla \times \nabla\psi &= 0 \\
\nabla \cdot (\nabla \times \mathbf{a}) &= 0 \\
\nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \\
\nabla \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \nabla\psi + \psi (\nabla \cdot \mathbf{a}) \\
\nabla \times (\psi \mathbf{a}) &= \nabla\psi \times \mathbf{a} + \psi (\nabla \times \mathbf{a}) \\
\nabla (\mathbf{a} \cdot \mathbf{b}) &= (\mathbf{a} \cdot \nabla)\, \mathbf{b} + (\mathbf{b} \cdot \nabla)\, \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \\
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\, \mathbf{a} - (\mathbf{a} \cdot \nabla)\, \mathbf{b} \\
\nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})
\end{aligned}$$

4. Let

$$\begin{aligned}
\boldsymbol{\omega} &= x\mathbf{d}x + y\mathbf{d}y + z\mathbf{d}z \\
\boldsymbol{\eta} &= \frac{x}{r}\mathbf{d}x + \frac{y}{r}\mathbf{d}y + \frac{z}{r}\mathbf{d}z
\end{aligned}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Find the value of the following expressions using differential forms:

$$\begin{aligned} \nabla \cdot \boldsymbol{\omega} \\ \nabla \times \boldsymbol{\omega} \\ \nabla \cdot \boldsymbol{\eta} \\ \nabla \times \boldsymbol{\eta} \end{aligned}$$

Also, show that for any vector $\mathbf{a}$,

$$(\mathbf{a} \cdot \nabla)\, \boldsymbol{\eta} = \frac{1}{r}\left[\mathbf{a} - \mathbf{n}\,(\mathbf{a} \cdot \mathbf{n})\right]$$