Exercises using the wedge product, exterior derivative and Hodge dual

1. Let $\boldsymbol{\omega}, \boldsymbol{\eta}$ be 1-forms in 3-dimensions. Show that the components of $\boldsymbol{\omega} \wedge \boldsymbol{\eta}$ are the same three functions as the covariant components of the cross product of the corresponding vectors.

2. Let $\boldsymbol{\tau}$ be a 2-form in 3-dimensions. Find $\boldsymbol{\tau} \wedge \boldsymbol{\omega}$ and $\boldsymbol{\tau} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\eta}$.

3. Explore the effect of the exterior derivative, $\mathbf{d}$, on 2-forms and on 3-forms. In particular, find

$$\mathbf{d}\boldsymbol{\omega}$$

   where

$$\boldsymbol{\omega} = \frac{1}{2!} v^i \varepsilon_{ijk} \mathbf{d}x^j \wedge \mathbf{d}x^k$$

4. Explicitly write the form of the generalized Stokes theorem for 2-forms in 3-dimensions.

5. Suppose

$$\begin{aligned} h &= h(x, y) \\ f &= f(x, y) \end{aligned}$$

   Consider the differential equation

$$\mathbf{d}f = h\mathbf{d}x + \mathbf{d}y$$

   What condition on the function h is necessary and sufficient to guarantee that this equation is integrable? Note that this single equation is equivalent to the system of equations

$$\begin{aligned} \frac{\partial f}{\partial x} &= h \\ \frac{\partial f}{\partial y} &= 1 \end{aligned}$$

6. Let f be a function of 3 variables, (x, y, z). For what values of the constants a and b is the equation

$$\mathbf{d}f = xy^2\mathbf{d}x + \left(ax^2y + byz^3\right)\mathbf{d}y + y^2z^2\mathbf{d}z$$

   integrable? Using those values, integrate to find the function f.

7. Let

$$\begin{aligned} f &\in \Lambda_0 \\ \boldsymbol{\omega} &\in \Lambda_1 \\ \boldsymbol{\eta} &\in \Lambda_2 \\ \boldsymbol{\tau} &\in \Lambda_3 \end{aligned}$$

Find the Hodge dual of each, ${}^*f, {}^*\boldsymbol{\omega}, {}^*\boldsymbol{\eta}, {}^*\boldsymbol{\tau}$ and the double dual of each, ${}^{**}f, {}^{**}\boldsymbol{\omega}, {}^{**}\boldsymbol{\eta}, {}^{**}\boldsymbol{\tau}$. You will need the relations,

$$
\begin{aligned}
e^{ijk} &= \frac{1}{\sqrt{g}}\varepsilon^{ijk} \\
e_{ijk} &= \sqrt{g}\varepsilon_{ijk}
\end{aligned}
$$

and the identity

$$
\begin{aligned}
e^{ijk}e_{ijk} &= 6\delta^{ijk}_{lmn} \\
&= 6\delta^{[i}_{l}\delta^{j}_{m}\delta^{k]}_{n} \\
&= \delta^i_l\delta^j_m\delta^k_n + \delta^j_l\delta^k_m\delta^i_n + \delta^k_l\delta^i_m\delta^j_n - \delta^j_l\delta^i_m\delta^k_n - \delta^i_l\delta^k_m\delta^j_n - \delta^k_l\delta^j_m\delta^i_n
\end{aligned}
$$

(Note: the constants have been corrected from the class notes. Check the trace of both sides:

$$
6 = 27 + 3 + 3 - 9 - 9 - 9)
$$