Exercises using the wedge product

1. Find the dimension of the space of p-forms, Λ_p, on a space of n-dimensions.

2. Find the dimension of the space of forms,

$$\dim\left(\Lambda_0 \otimes \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_n\right)$$

   This is just the sum of dimensions of the Λ_p,

$$\sum_{p=0}^{n} \dim\left(\Lambda_p\right)$$

3. We may define the wedge product,

$$\wedge : \Lambda_1 \otimes \Lambda_1 \to \Lambda_2$$

   as an antisymmetric, linear, and associative mapping given by antisymmetrizing the outer product:

$$\boldsymbol{\omega} \wedge \boldsymbol{\eta} = \frac{1}{2}\left(\boldsymbol{\omega} \otimes \boldsymbol{\eta} - \boldsymbol{\eta} \otimes \boldsymbol{\omega}\right)$$

   Use these properties on the pairwise product to show that if $\boldsymbol{\omega}$ is a p-form ($\boldsymbol{\omega} \in \Lambda_p$)and $\boldsymbol{\eta}$ is a q-form ($\boldsymbol{\eta} \in \Lambda_q$), then

$$\boldsymbol{\omega} \wedge \boldsymbol{\eta} = (-1)^{pq}\, \boldsymbol{\eta} \wedge \boldsymbol{\omega}$$

4. Let $n = 3$. Let two 1-forms be given by

$$\begin{aligned} \boldsymbol{\omega} &= \omega_i \mathbf{d}x^i \\ \boldsymbol{\eta} &= \eta_j \mathbf{d}x^j \end{aligned}$$

   Write out the wedge product of these two 1-forms explicitly. Compare the result to the components as the cross product of the corresponding covariant vectors.