1 A linear representation of the conformal group

We wish to find a linear representation of the conformal group.

On flat spacetime, in Cartesian coordinates, we can represent the action of the conformal group as the set of transformations

$$\begin{aligned} \tilde{x}^a &= \Lambda^a_b x^b \\ \tilde{x}^a &= x^a + b^a \\ \tilde{x}^a &= \lambda x^a \\ \tilde{x}^a &= \frac{x^a + x^2 c^a}{1 + 2c_a x^a + c^2 x^2} \end{aligned}$$

where

$$\eta_{cd} \Lambda^c_a \Lambda^d_b = \eta_{ab}$$

so that the Λ^a_b are Lorentz transformations. The second transformation is a translation, the third a dilatation and the final transformations, called *special conformal transformations* may be seen to be conformal by considering the *inversion*,

$$\tilde{x}^a = \frac{x^a}{x^2}$$

Inversion is a discrete transformation, and since

$$\tilde{s}^2 = \frac{x^a x_a}{(x^2)^2} = \frac{1}{(x^2)^2} s^2$$

it is conformal. A special conformal transformation is an inversion followed by translation, followed by a second inversion:

$$x^a \rightarrow \frac{x^a}{x^2} \rightarrow \frac{x^a}{x^2} + c^a \rightarrow \frac{\frac{x^a}{x^2} + c^a}{\left(\frac{x^b}{x^2} + c^b\right)^2} = \frac{x^a + x^2 c^a}{1 + 2c_a x^a + c^2 x^2}$$

Notice that there are 15 conformal transformations: 6 Lorentz, 4 translation, 1 dilatation and 4 special conformal transformations.

We define the conformal transformations as those transformations preserving the light cone. This is equivalent to preserving angles, and also equivalent to preserving ratios of lengths. Since a light cone centered at a point a^b may be described by the equation

$$\begin{aligned} 0 &= \lambda \left(x^b - a^b\right)(x_b - a_b) \\ &= \lambda x^2 - 2\lambda x^b a_b + \lambda a^2 \end{aligned}$$

conformal transformations will be those transformations which map

$$\left(\lambda, a^b\right) \rightarrow \left(\sigma, b^b\right)$$

Label these parameters as

$$\begin{aligned} B^b &= \lambda a^b \\ B^4 &= \lambda \\ B^5 &= \frac{1}{2}\lambda a^2 \end{aligned}$$

Then $B^A = \left(B^a, B^4, B^5\right)$ satisfies

$$B^a B_a - 2B^4 B^5 = 0$$

where $B^a B_a = \eta_{ab} B^a B^b$. Define a 6-dim metric,

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix}$$

so that

$$\eta_{AB}B^{A}B^{B}=0$$

Any transformation preserving this null quadratic form induces a change in $\left(\lambda,a^{b}\right)$ that maps to another 4-dim light cone.

Although we want η_{AB} in this particular form, we notice that if we were to diagonalize it the lower right corner becomes $\left(\begin{array}{cc}\pm1 & 0\\ 0 & \mp1\end{array}\right)$, so η_{AB} as four positive and two negative eigenvalues. The linear transformations preserving η_{AB} therefore comprise the group $O\left(4,2\right)$. Since we may recover coordinates, a^{b}, near the origin and coordinatesm $\frac{a^{b}}{a^{2}}$, near infinity, on Minkowski space by forming the ratios

$$\begin{aligned}a^{b} &= \frac{B^{b}}{B^{4}}\\ \frac{a^{b}}{a^{2}} &= \frac{B^{b}}{2B^{5}}\end{aligned}$$

we see that any scaling, $B^{A}\rightarrow\alpha B^{A}$ has no effect on spacetime. We therefore restrict to $SO\left(4,2\right)$. Finally, note that

$$a^{2}=\frac{2B^{5}}{B^{4}}$$

2 The special orthogonal group, SO(4,2)

Next, we find the generators of SO(4,2). Let $g=1+\varepsilon$ be infinitesimally near the identity and require

$$\begin{aligned}\eta_{AB} &= \eta_{CD}g_{\ A}^{C}g_{\ B}^{D}\\ &= \eta_{CD}\left(\delta_{A}^{C}+\varepsilon_{\ A}^{C}\right)\left(\delta_{B}^{D}+\varepsilon_{\ B}^{D}\right)\\ &= \eta_{AB}+\eta_{CB}\varepsilon_{\ A}^{C}+\eta_{AD}\varepsilon_{\ B}^{D}+O\left(\varepsilon^{2}\right)\end{aligned}$$

and therefore, using the metric to lower the upper index on ε,

$$\varepsilon_{BA}=-\varepsilon_{AB}$$

as we have found before for pseudo-orthogonal groups. We divide the antisymmetric matrices into four types:

$$\begin{aligned}\left[M_{ab}\right] &= \left(\begin{array}{cc}\varepsilon_{ab} & \\ & 0\end{array}\right)\\ \left[P_{a}\right] &= \left(\begin{array}{cccccc}0 & & & & -1 & \\ & 0 & & & 1 & \\ & & 0 & & 1 & \\ & & & 0 & 1 & \\ 1 & -1 & -1 & -1 & 0 & \\ & & & & & 0\end{array}\right)\\ \left[K_{a}\right] &= \left(\begin{array}{cccccc}0 & & & & & -1\\ & 0 & & & & 1\\ & & 0 & & & 1\\ & & & 0 & & 1\\ & & & & 0 & \\ 1 & -1 & -1 & -1 & & 0\end{array}\right)\\ \left[D\right] &= \left(\begin{array}{cccccc}0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & -1\\ & & & & 1 & 0\end{array}\right)\end{aligned}$$

The first of these generate Lorentz transformation, $\Lambda^a_{\ b} = \exp\left(\frac{1}{2} w^a_{\ b} \varepsilon^b_{\ a}\right)$, where $\varepsilon^b_{\ a} = \eta^{bc}\varepsilon_{ca} = -\eta^{bc}\varepsilon_{ac}$, and we will say no more about them. We raise the first index on the remaining generators to recover $\varepsilon^A_{\ B}$,

$$
\begin{aligned}
[P_a]^A_{\ \ B} &= \begin{pmatrix} \eta_{ab} & & \\ & 0 & -1 \\ & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & & & & -1 & \\ & 0 & & & 1 & \\ & & 0 & & 1 & \\ & & & 0 & 1 & \\ 1 & -1 & -1 & -1 & 0 & \\ & & & & & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & & & & 1 & \\ & 0 & & & 1 & \\ & & 0 & & 1 & \\ & & & 0 & 1 & \\ & & & & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \\
[K_a]^A_{\ \ B} &= \begin{pmatrix} 0 & & & & & 1 \\ & 0 & & & & 1 \\ & & 0 & & & 1 \\ & & & 0 & & 1 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ & & & & 0 & 0 \end{pmatrix} \\
[D]^A_{\ \ B} &= \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & -1 & \\ & & & & & 1 \end{pmatrix}
\end{aligned}
$$

We exponentiate each in turn.

For P_a,

$$
\begin{aligned}
g\left(b^a\right) &= \exp\left(b^a P_a\right) \\
&= \exp \begin{pmatrix} 0 & & & & b^0 & \\ & 0 & & & b^1 & \\ & & 0 & & b^2 & \\ & & & 0 & b^3 & \\ & & & & 0 & 0 \\ -b^0 & b^1 & b^2 & b^3 & 0 & 0 \end{pmatrix} \\
&= 1 + \begin{pmatrix} 0 & & & & b^0 & \\ & 0 & & & b^1 & \\ & & 0 & & b^2 & \\ & & & 0 & b^3 & \\ & & & & 0 & 0 \\ -b^0 & b^1 & b^2 & b^3 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & 0 \\ & & & & b^a b_a & 0 \end{pmatrix}
\end{aligned}
$$

with the remaining powers vanishing. Therefore, the four transformations give

$$
\tilde{B}^A = \begin{pmatrix} 1 & & & & b^0 & \\ & 1 & & & b^1 & \\ & & 1 & & b^2 & \\ & & & 1 & b^3 & \\ & & & & 1 & 0 \\ -b^0 & b^1 & b^2 & b^3 & \frac{1}{2} b^a b_a & 1 \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \\ B^4 \\ B^5 \end{pmatrix}
$$

$$= \begin{pmatrix} B^0 + b^0 B^4 \\ B^1 + b^1 B^4 \\ B^2 + b^2 B^4 \\ B^3 + b^3 B^4 \\ B^4 \\ B^5 + B^a b_a + \frac{1}{2} B^4 b^a b_a \end{pmatrix}$$

On spacetime, this transformation of B^A gives

$$\begin{aligned} \tilde{x}^a &= \frac{\tilde{B}^a}{\tilde{B}^4} \\ &= \frac{B^a + b^a B^4}{B^4} \\ &= x^a + b^a \end{aligned}$$

and is therefore a translation. On a coordinate centered on infinity, this same translation becomes

$$\begin{aligned} \tilde{y}^a &= \frac{\tilde{x}^a}{\tilde{x}^2} \\ &= \frac{\tilde{B}^a}{2\tilde{B}^5} \\ &= \frac{B^a + b^a B^4}{2\left(B^5 + B^a b_a + \frac{1}{2} B^4 b^a b_a\right)} \\ &= \frac{\frac{B^a}{2B^5} + b^a \frac{B^4}{2B^5}}{1 + \frac{B^a b_a}{B^5} + \frac{B^4 b^a b_a}{2B^5}} \\ &= \frac{y^a + \frac{B^4}{2B^5} b^a}{1 + 2y^a b_a + \frac{B^4}{2B^5} b^a b_a} \end{aligned}$$

and since $x^2 = \frac{2B^5}{B^4} = \frac{1}{y^2}$,

$$\begin{aligned} \tilde{y}^a &= \frac{y^a + \frac{B^4}{2B^5} b^a}{1 + 2y^a b_a + \frac{B^4}{2B^5} b^a b_a} \\ &= \frac{y^a + y^2 b^a}{1 + 2y^a b_a + y^2 b^2} \end{aligned}$$

This means that in a neighborhood of the point at infinity, a translation takes a somewhat complicated form – the same form that a special conformal transformation takes in a neighborhood of the origin.

Now consider a special conformal transformation. We have

$$\begin{aligned} g(c^a) &= \exp(c^a K_a) \\ &= \exp \begin{pmatrix} 0 & & c^a \\ c_a & 0 & 0 \\ & 0 & 0 \end{pmatrix} \\ &= 1 + \begin{pmatrix} 0 & & c^a \\ c_a & 0 & 0 \\ & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_a c^a \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & c^a \\ c_a & 1 & \frac{1}{2} c_a c^a \\ & 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{B}^A &= \begin{pmatrix} 1 & & c^a \\ c_a & 1 & \frac{1}{2}c_a c^a \\ & 0 & 1 \end{pmatrix} \begin{pmatrix} B^a \\ B^4 \\ B^5 \end{pmatrix} \\
&= \begin{pmatrix} B^a + B^5 c^a \\ B^4 + B^a c_a + \frac{1}{2}c^2 B^5 \\ B^5 \end{pmatrix}
\end{aligned}$$

The coordinates near the origin and near infinity change according to:

$$\begin{aligned}
\tilde{x}^a &= \frac{\tilde{B}^a}{\tilde{B}^4} \\
&= \frac{B^a + B^5 c^a}{B^4 + B^a c_a + \frac{1}{2}c^2 B^5} \\
&= \frac{\frac{B^a}{B^4} + \frac{B^5}{B^4}c^a}{1 + \frac{B^a}{B^4}c_a + \frac{1}{2}c^2\frac{B^5}{B^4}} \\
&= \frac{x^a + \frac{1}{2}x^2 c^a}{1 + x^a c_a + \frac{1}{4}c^2 x^2}
\end{aligned}$$

Now redefine the parameter, replacing $c^a \rightarrow 2c^a$ and we have

$$\tilde{x}^a = \frac{x^a + x^2 c^a}{1 + 2x^a c_a + c^2 x^2}$$

For the inverse coordinate,

$$\begin{aligned}
\tilde{y}^a &= \frac{\tilde{x}^a}{\tilde{x}^2} \\
&= \frac{\tilde{B}^a}{2\tilde{B}^5} \\
&= \frac{B^a + B^5 c^a}{2B^5} \\
&= y^a + \frac{1}{2}c^a
\end{aligned}$$

and making the same change of parameter, we have a simple translation at infinity,

$$\tilde{y}^a = y^a + c^a$$

Finally, we find the dilatations,

$$\begin{aligned}
g(\lambda) &= \exp(\lambda D) \\
&= \exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & e^{\lambda} \end{pmatrix}
\end{aligned}$$

The 6-vector changes to

$$\begin{aligned}
\tilde{B}^A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & e^{\lambda} \end{pmatrix} \begin{pmatrix} B^a \\ B^4 \\ B^5 \end{pmatrix} \\
&= \begin{pmatrix} B^a \\ e^{-\lambda}B^4 \\ e^{\lambda}B^5 \end{pmatrix}
\end{aligned}$$

so the spacetime coordinates transform as

$$\begin{aligned}\tilde{x}^a &= \frac{\tilde{B}^a}{\tilde{B}^4}\\ &= e^{\lambda}\frac{B^a}{B^4}\\ &= e^{\lambda}x^a\end{aligned}$$

and

$$\begin{aligned}\tilde{y}^a &= \frac{\tilde{B}^a}{2\tilde{B}^5}\\ &= e^{-\lambda}\frac{B^a}{2B^5}\\ &= e^{-\lambda}y^a\end{aligned}$$

Notice that the coordinates at the origin and at infinity transform oppositely under dilatations. This will have important consequences.

3 Maurer-Cartan structure equations for SO(4,2)

We already know that the Lie algebra for pseudo-orthogonal groups takes the form

$$[M_{AB}, M_{CD}] = -\frac{1}{2}(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC})$$

where I've written the labels down so that the antisymmetrizations, $[A,B]$ and $[C,D]$ are easier to see. Defining a dual basis of 1-forms by

$$\left\langle M^A_{\ B}, \omega^C_{\ D}\right\rangle = 2\Delta^{AC}_{DB}$$

the Maurer-Cartan structure equations are

$$\mathbf{d}\omega^A_{\ B} = \omega^C_{\ B}\wedge\omega^A_{\ C}$$

where the the labels have been raised with η^{AB}.

When we use the metric in the off-diagonal form with nonzero elements $(\eta_{ab}, \eta_{45}, \eta_{54})$, the antisymmetry relations give some unexpected relations. We have

$$\omega^A_{\ B} = -\eta^{AD}\eta_{BC}\omega^C_{\ D}$$

We want to separate the 6-dim indices, A, B, into constituent parts, $A = (a, 4, 5)$. Therefore, for $A = a$,

$$\begin{aligned}\omega^a_{\ b} &= -\eta^{aD}\eta_{bC}\omega^C_{\ D}\\ &= -\eta^{ad}\eta_{bc}\omega^c_{\ d}\\ \omega^a_{\ 4} &= -\eta^{aD}\eta_{4C}\omega^C_{\ D}\\ &= -\eta^{ad}\eta_{45}\omega^5_{\ d}\\ &= \eta^{ad}\omega^5_{\ d}\\ \omega^a_{\ 5} &= -\eta^{aD}\eta_{5C}\omega^C_{\ D}\\ &= -\eta^{ad}\eta_{54}\omega^4_{\ d}\\ &= \eta^{ad}\omega^4_{\ d}\end{aligned}$$

Next, setting $A = 4$,

$$\begin{aligned}\omega^4_{\ b} &= -\eta^{4D}\eta_{bC}\omega^C_{\ D}\\ &= -\eta^{45}\eta_{bc}\omega^c_{\ 5}\\ &= \eta_{bc}\omega^c_{\ 5}\end{aligned}$$

$$\begin{aligned}
\omega^4_4 &= -\eta^{4D}\eta_{4C}\omega^C_D \\
&= -\eta^{45}\eta_{45}\omega^5_5 \\
&= -\omega^5_5 \\
\omega^4_5 &= -\eta^{4D}\eta_{5C}\omega^C_D \\
&= -\eta^{45}\eta_{54}\omega^4_5 \\
&= -\omega^4_5 \\
&= 0
\end{aligned}$$

Finally, with $A = 5$, we have

$$\begin{aligned}
\omega^5_b &= \eta_{bc}\omega^c_4 \\
\omega^5_4 &= -\omega^5_4 \\
&= 0 \\
\omega^5_5 &= -\omega^4_4
\end{aligned}$$

Notice that the connection forms with one or two $A = 5$ indices may be expressed in terms of corresponding $A = 4$ forms. An independent set is given by $\left\{\omega^a_b, \omega^4_b, \omega^a_4, \omega^4_4\right\}$, where we eliminate the rest using

$$\begin{aligned}
\omega^a_5 &= \eta^{ab}\omega^4_b \\
\omega^5_a &= \eta_{ab}\omega^b_4 \\
\omega^5_5 &= -\omega^4_4 \\
\omega^5_4 &= 0
\end{aligned}$$

We can simplify the names of the remaining independent set by dropping the "4" indices,

$$\left\{\omega^a_b, \omega^4_b, \omega^a_4, \omega^4_4\right\} \to \left\{\omega^a_b, \omega_b, \omega^a, \omega\right\}$$

Another efficiency measure is to assume the wedge product between forms,

$$\omega^C_B \wedge \omega^A_C \to \omega^C_B \omega^A_C$$

so that whenever two forms are written next to each other, the wedge product is assumed.

Employing these simplifications, we make the same breakdown of the structure equations, $\mathbf{d}\omega^A_B = \omega^C_B\omega^A_C$. For the connection forms dual to Lorentz transformations we find that

$$\begin{aligned}
\mathbf{d}\omega^a_b &= \omega^C_b\omega^a_C \\
&= \omega^c_b\omega^a_c + \omega^4_b\omega^a_4 + \omega^5_b\omega^a_5 \\
&= \omega^c_b\omega^a_c + \omega^4_b\omega^a_4 + \omega^5_b\omega^a_5 \\
&= \omega^c_b\omega^a_c + \omega^4_b\omega^a_4 + \eta_{bd}\omega^d_4\eta^{ac}\omega^4_c \\
&= \omega^c_b\omega^a_c + \left(\delta^a_d\delta^c_b - \eta_{bd}\eta^{ac}\right)\omega^4_c\omega^d_4 \\
&= \omega^c_b\omega^a_c + 2\Delta^{ac}_{db}\omega^4_c\omega^d_4 \\
&= \omega^c_b\omega^a_c + 2\Delta^{ac}_{db}\omega_c\omega^d
\end{aligned}$$

For the connection forms dual to translations, we have

$$\begin{aligned}
\mathbf{d}\omega^a_4 &= \mathbf{d}\omega^a \\
&= \omega^C_4\omega^a_C \\
&= \omega^c_4\omega^a_c + \omega^4_4\omega^a_4 \\
&= \omega^c\omega^a_c + \omega\omega^a
\end{aligned}$$

The special conformal transformations are similar,

$$\begin{aligned}
\mathbf{d}\omega^4_a &= \mathbf{d}\omega_a \\
&= \omega^C_a\omega^4_C \\
&= \omega^c_a\omega_c + \omega_a\omega
\end{aligned}$$

Finally, the dilatation equation is

$$\begin{aligned}\mathbf{d}\omega_4^4 &= \mathbf{d}\omega \\ &= \omega_4^C \omega_C^4 \\ &= \omega^a \omega_a\end{aligned}$$

Collecting the results, we have the Maurer-Cartan structure equations for the conformal group, $SO(4,2)$:

$$\begin{aligned}\mathbf{d}\omega^a_{\ b} &= \omega^c_{\ b}\omega^a_{\ c} + 2\Delta^{ac}_{db}\omega_c\omega^d \\ \mathbf{d}\omega^a &= \omega^c\omega^a_{\ c} + \omega\omega^a \\ \mathbf{d}\omega_a &= \omega^c_{\ a}\omega_c + \omega_a\omega \\ \mathbf{d}\omega &= \omega^a\omega_a\end{aligned}$$

Consider these for a moment. The Lorentz transformations include the same terms, $\mathbf{d}\omega^a_{\ b} = \omega^c_{\ b}\omega^a_{\ c}$, that we had from the Poincaré gauging, but there is an additional term, $2\Delta^{ac}_{db}\omega_c\omega^d$, that exists only because we have *both* the translations and the special conformal translations. The equation for translations also has one new term. The first term on the right, $\omega^c\omega^a_c$, corrects for the effect of a local Lorentz transformation on the derivative of the solder form, ω^a, while the second term, $\omega\omega^a$, makes the translation equation invariant under local changes of units. The third equation is much like the second. Notice that if we write the fields in the same order as for the translation equation, that is,

$$\begin{aligned}\mathbf{d}\omega^a &= \omega^c\omega^a_{\ c} + \omega\omega^a \\ \mathbf{d}\omega_a &= -\omega_c\omega^c_{\ a} - \omega\omega_a\end{aligned}$$

that they have opposite signs for the terms on the right. That means that a local Lorentz transformation that transforms the solder form in one way will transform the special conformal gauge field in the opposite way; a dilatation that transforms the solder form by e^{λ} will transform the special conformal gauge field by $e^{-\lambda}$. Finally, consider the dilatation equation. The Weyl vector no longer has vanishing curl, $\mathbf{d}\omega$. Depending on how we carry out the gauging of the conformal group, the new term can be important.