Quantum mechanics as a measurement theory on biconformal space

By Lara B. Anderson and James T. Wheeler
Department of Physics, Utah State University

September 26, 2005

Abstract

Biconformal spaces contain the essential elements of quantum mechanics, making the independent imposition of quantization unnecessary. Based on three postulates characterizing motion and measurement in biconformal geometry, we derive standard quantum mechanics, and show how the need for probability amplitudes arises from the use of a standard of measurement. Additionally, we show that a postulate for unique, classical motion yields Hamiltonian dynamics with no measurable size changes, while a postulate for probabilistic evolution leads to physical dilatations manifested as measurable phase changes. Our results lead to the Feynman path integral formulation, from which follows the Schrödinger equation. We discuss the Heisenberg uncertainty relation and fundamental canonical commutation relations.

1 Introduction

In this work, we show that conformal symmetry gives rise to the essential elements of quantization. Gauging the conformal group, we produce a manifold that is naturally equipped with symplectic structure, metric structure, and scale invariance. Formulating a measurement theory consistent with these structures, we then demonstrate a subclass of solution manifolds for which the measurement theory describes quantum mechanics.

Much recent work has shown that the combination of a symplectic manifold with a Riemannian metric – often realized elegantly as a Kähler manifold

– provides a sufficient background to allow quantization. We briefly review some of these findings below. Additionally, we summarize some work on scale invariance that reveals close parallels between dilatation factors and quantum states. Then in the body of the paper, we show, first, that these three factors occur naturally in biconformal spaces, second, that we can formulate a measurement theory consistent with all three factors, and finally, that on some solution manifolds the standard formulation of quantum mechanics follows automatically.

In recent years, the relationship between symplectic structure and metric structure on manifolds has been a subject of considerable interest. The interplay between these two fundamental structures underlies both classical and quantum behavior, providing insights into the very nature of quantization. In classical mechanics, the presence of a symplectic form defines the characteristics of a classical Newtonian or relativistic phase space, while the Euclidean metric is required to make the interpretation of Hamilton's equations unambiguous [1]. Phase space, of course, is essential for the formulation of quantum theory. All of the standard quantization approaches, including canonical quantization [2], phase space quantization [3], geometric quantization [4], and path integral quantization [5], embody the uncertainty principle, and therefore utilize the fundamental coupling between position and momentum that is characteristic of phase space and defined by the symplectic form. The work of Berezin [6] and Klauder [1] further establishes that imposing a Riemannian metric on classical phase space is sufficient to support quantization. Following along these lines, recent work has shown that the Darboux theorem for symplectic manifolds can also play a role in quantum behavior. Isidro [7], [8] demonstrates that there always exists a coordinate transformation (similar to Darboux coordinates) that transforms a quantum system into the semiclassical regime. That is, the quantum system is transformed into a system that can be studied by means of a perturbation in powers of h around a certain local vacuum.

The relationship of dilatational symmetry to quantum theory was first studied by London. In [9], London showed that dilatation factors of the form $e^{\alpha\phi}$ replicate the behavior of the Schrödinger wave function if the constant parameter α is pure imaginary. This finding reflects the close relationship between solutions to the Schrödinger equation and the Fokker-Planck equation. This relationship was utilized in a diffusive theory formulated by one of us (JTW) [10], where dilatational symmetry is used to write a theory of quantum measurement. A model for the interpretation of spacetime as a

Weyl geometry was proposed, based on the hypothesis that a system moves on any given path with a probability which is inversely proportional to the resulting change in length of the system. Consistent with London's result, these probabilities were shown to be the Green's functions for a diffusion equation instead of describing unitary evolution. Nonetheless, the solutions of this diffusion equation are identical to the stationary state solutions of the Schrödinger equation if the line integral of the Weyl field equals the action functional divided by $\hbar$. The central results indicated that the presence of dilatational symmetry could create inherently quantum behavior in a physical system. The theory developed was a consistent, stochastic model of quantum mechanics resembling Nelson's theory, which formulates quantum mechanics as a classical conservative diffusion process [11]. Since [10] predicts that measurable probabilities involve the product of amplitudes, the theory resolves the locality issues of Nelson's formulation.

All three of these significant structures – symplectic, Riemannian metric, and scale invariance – arise naturally within a particular gauging of the conformal group. The *biconformal* gauging of the conformal group produces an 8- or $2n$-dim manifold with local Lorentz and dilatational symmetry (scale invariance). The spaces also possess a symplectic form. Because biconformal gauging preserves the involutive automorphism of conformal weight interchange ([12]-[14]), fields naturally occur in conformally conjugate pairs. After gauging, these conjugate pairs become canonically conjugate with respect to a natural symplectic form given by $\mathbf{d}\omega$, where ω is the gauge field of dilatations. Finally, biconformal gauge theory, unlike Poincaré gauge theory, provides a natural metric since the structure constants of the conformal algebra produce a non-degenerate group-invariant Killing metric, K_{AB}.

We will show that the interactions between the dilatational symmetry, metric structure and symplectic structure of biconformal space give rise to a model that uniquely incorporates classical and quantum behavior as a natural consequence of a geometry – without independent quantization. By defining measurement in a conformal gauge theory and utilizing the fundamental structure, we reproduce a formulation of standard quantum theory. To accomplish this, it is of central importance to note that the biconformal field and structure equations admit both real and complex solutions. Combining a complex solution of the field equations with a gauge in which the Weyl vector is imaginary, we show that transplantation of Lorentz invariant lengths produces a phase factor rather than a proper dilatation. Furthermore, since measurement in a scale-invariant geometry requires the use of a standard, we

show that meaningful predictions require a product of conjugate quantities, an idea corresponding to the product of a wave function with its conjugate in quantum mechanics. Finally, postulating a probabilistic law of motion based on physical size changes, we derive a product of Feynman path integrals as a measurable prediction of the model. These results are largely due to the use of biconformal solutions in which a certain involutive automorphism is manifested as complex conjugation. This guarantees non-integrability of spacetime curves, and as we show, results in a bracket relation closely related to the quantum commutator.

In addition to a potential arena for quantum mechanics, our choice of the biconformal gauging of the conformal group is motivated by its strength as a gravity model. In the past, gaugings of the conformal group to create a theory of gravity have given rise to unphysical size changes and the presence of ghosts. However, these problems are resolved ([15]-[20]) by choosing the relativistic homothetic group as the local symmetry of the model (a biconformal gauging), which leads to a conformal gravity theory on a $2n$-dimensional manifold. In addition to its symplectic structure, this theory possesses torsion-free spaces consistent with general relativity and electromagnetism. The doubled dimension of the base manifold results in a dimensionless volume element, making it possible to write invariant actions linear in the curvatures [16]. Wehner and Wheeler demonstrated that the resulting field equations lead to the Einstein equation and hence, general relativity, on an n-dimensional submanifold. Also, the biconformal gauging can be extended to the superconformal group to form a new class of supergravity models [17], which not only produce the Einstein field equations, but Dirac and Rarita-Schwinger type equations for fermionic fields as well. Observing the success of these gravity and supergravity models it is natural to inquire whether the additional structures supplied by the conformal group might hold some further physical significance. This paper is an attempt to gain insight into this question. While we do not attempt to formulate quantum gravity in this work, the simultaneous existence of quantum behavior and gravity within biconformal spaces gives the hope that this theory could lead to a novel approach to the subject.

In the next three Sections of this work we introduce our basic postulates, each of which deals with a conceptually distinct piece of the physical model. The first postulate describes the geometric arena for our considerations, while issues of measurement are handled by the second. Finally, we require a postulate governing dynamical evolution. We introduce two possible choices for

this third postulate. The first of these (3A) leads directly to a description of biconformal space as a gauge formulation of classical Hamiltonian dynamics. Our second choice for the third postulate (3B) describes a probabilistic evolution of a system which is dependent upon measurable dilatation. It is this postulate that leads us to a formulation of standard unitary quantum mechanics. Section 5 comprises a brief summary of our results.

2   The physical arena

The space in which we live, act and measure was postulated by Newton to be 3-dimensional and Euclidean. By the early years of the last century, the dimension was increased to four, and within another decade the geometry became Riemannian. With the advent of quantum mechanics, it gradually became clear that other spaces – phase space and Hilbert space – play essential roles in our understanding of the world. Even more dramatically, Kaluza-Klein field theories and string theory have, over the last fifty years, been formulated in dimensions as high as 26, while other geometries as diverse as lattices and spin foam have been used to study quantum gravity.

We seek a physical arena which contains 4-dim spacetime in a straightforward way, but which is large enough and structured enough to contain all known physics. This arena should follow from fundamental principles. Since all known fundamental interactions are described by gauge theories based on Lie groups, we seek a gauge theory of a fundamental symmetry. To choose the symmetry, our best guide is to observe the constraints placed upon physical models by experiment. Specifically, we note that laboratory measurements are necessarily unaffected by rotating, boosting, or displacing our experimental apparatus. Moreover, every measurement requires the use of a standard to be meaningful. Measured magnitudes must be expressed as dimensionless ratios. Examples of this include measuring time relative to the oscillations of Cesium in atomic clocks or measuring masses in MeV. We therefore demand invariance under scalings (i.e. choice of units), global Lorentz transformations and translations (see postulates 2 & 3, below). The Lie group characterizing these invariances is the conformal group ($O\left(4,2\right)$ or its covering group, $SU(2,2)$). We now require a gauge theory of the conformal group.

Conformal gauge theories have been studied extensively. Discussions of these are provided in ([13]-[20]). We write the generators of the conformal

5

group as follows: Lorentz transformations, $M_b^a = -M_{ba} = \eta_{ac}M_b^c$; translations, P_a; special conformal transformations, K^a; and dilatations, D; where $a, b, \ldots = 0, 1, 2, 3$. The commutation relations of these generators are given in Appendix A, and a discussion of certain representations is given in Appendix B. Among the properties of conformal symmetry reviewed in Appendix B, we note the existence of two involutive automorphisms of the conformal algebra [13],[14]. The first acts on the generators according to:

$$\sigma_1 : (M_b^a, P_a, K^a, D) \to (M_b^a, -P_a, -K^a, D)$$

and identifies the residual local Lorentz and dilatation symmetry characteristic of biconformal gauging. This involutive automorphism corresponds to

$$\sigma_1 : (M_b^a, P_a) \to (M_b^a, -P_a)$$

for the Poincaré Lie algebra, or

$$\sigma_1 : (M_b^a, P_a, D) \to (M_b^a, -P_a, D)$$

for the Weyl algebra. However, the conformal group admits a second involution that is not possible in the Poincaré or Weyl cases, namely,

$$\sigma_2 : (M_b^a, P_a, K^a, D) \to (M_b^a, K_a, P^a, -D)$$

This automorphism interchanges translations and special conformal transformations while inverting the conformal weight of dilatations.

Some representations of the conformal algebra, notably $su(2, 2)$, are necessarily complex. In such complex representations σ_2 can be realized as complex conjugation. Specifically, suppose we can find a representation in which P_a and K_a are complex conjugates, while M_b^a is real and D pure imaginary. Then σ_2 is equivalent to complex conjugation. Representations (and/or choices of basis within a representation) with this property will be called σ_C -representations and biconformal spaces for which the connection 1-forms (and hence curvatures) have this property will be called σ_C-spaces. Examples of σ_C-representations are given in Appendix B. It is important to notice that σ_C-representations do not exist for geometries based on the Poincaré or Weyl groups.

In this work, we use the biconformal gauging of the conformal group [15]. This choice automatically provides us with an arena containing symplectic,

metric, and scale invariant structures. Furthermore, as we will show, the σ_C-representations give these structures a form consistent with unitary evolution. We therefore postulate,

Postulate 1: The physical arena for quantum and classical physics is a σ_C biconformal space.

As noted in the introduction, biconformal gauging of the conformal group provides three properties that prove essential in constructing a quantum theory: symplectic structure, a Riemannian metric, and scale invariance. We discuss each in turn.

Symplectic structure, is present in all known solutions to the biconformal field equations. We note such a symplectic biconformal manifold may provide the ideal background for describing quantum phenomena, since the uncertainty principle makes the need for a phase-space-like formulation clear. The symplectic form arises as follows. Gauging D introduces a single gauge 1-form, ω, called the Weyl vector. The corresponding dilatational curvature 2-form is given by

$$\mathbf{\Omega} = \mathbf{d}\omega - 2\omega^a \omega_a, \tag{1}$$

where ω^a, ω_a (note that $\omega_a = \eta_{ab}\overline{\omega}^b$ for σ_C-reps) are 1-form gauge fields of the translation and special conformal transformations, respectively, which span the 8-dimensional space mentioned above as an orthonormal basis. In this work, differential forms are written in boldface, the overbar denotes complex conjugation, and the standard wedge product is implied in all multiplication of differential forms, $\omega\pi \equiv \omega \wedge \pi$.

For all torsion-free solutions to the biconformal field equations, the dilatational curvature takes the form,

$$\mathbf{\Omega} = \kappa \omega^a \omega_a \tag{2}$$

with κ constant, so the structure equation becomes,

$$\mathbf{d}\omega = (\kappa + 2)\,\omega^a \omega_a. \tag{3}$$

As a result, since ω^a, ω_a span the space, $\mathbf{d}\omega$ is manifestly closed and non-degenerate, hence a symplectic form.

The second important consequence of conformal symmetry, as noted in the introduction, is the biconformal metric. The metric arises from the group invariant Killing metric,

$$K_{\Sigma\Pi} = c_{\Delta\Sigma}{}^{\Lambda}c_{\Lambda\Pi}{}^{\Delta}, \tag{4}$$

where $c_{\Delta\Sigma}{}^{\Lambda}$ ($\Sigma, \Pi, \ldots = 1, 2, \ldots, 15$) are the (real) structure constants from the Lie algebra. Unlike the Killing metric of the Poincaré or Weyl groups, this metric has a nondegenerate projection to the base manifold. The 8-dim biconformal manifold spanned by P_a and K^a therefore has a natural pseudo-Riemannian metric. With the structure constants as in Appendix A, Eq.(49), this projection takes the form

$$K_{AB} = \begin{pmatrix} & \eta_{ab} \\ \eta_{ab} & \end{pmatrix}.$$

where $A, B = 0, 1, \ldots, 7$. Notice that K_{AB} has zero signature.

Finally, we note that biconformal spaces possess local scale invariance, and the structure equations and field equations following from the linear bosonic and supersymmetric actions both admit σ_C-solutions.

3 Measurement in biconformal geometry

In building any physical model, it is important to be clear about the relationship between the geometry and physical measurements. If biconformal space is to supply a successful model for physical processes, a necessary first step is to discuss the properties a theory of measurement must possess. In particular, the presence of a non-vanishing Weyl vector in these spaces leads to differences from Newtonian or relativistic measurement theories. When the dilatational curvature built from the curl of the Weyl vector does not vanish, vector lengths change nonintegrably about the manifold and we require an equivalence class of metrics rather than a unique metric. As Einstein observed of Weyl's attempt to form a theory of electromagnetism using scale-invariant geometry [21], the presence of dilatation curvature can create unphysical size changes. Any theory embodying scale invariance must address the consistency between any occurrence of physical size change and experimental results.

To discuss the effect of dilatations on physical fields, it is useful to define conformal weights. Given a set of objects on which the conformal group acts, we can define the subclass of definite conformal weight (or definite scaling weight) objects. The conformal weight, w, of a definite weight field, F, is then given by

$$D_\varphi : F \to e^{w\varphi}F. \tag{5}$$

where the transformation, D_φ, is a dilatation by a positive function, $\exp\varphi$. In the presence of a generic Weyl field, tensors of nonzero or indefinite conformal weight acquire values dependent upon their history. The resulting difficulties experienced in performing measurements in Weyl geometries are outlined in [10]. However, measurement may be unambiguously defined for objects of vanishing conformal weight. We therefore demand the same postulate for biconformal space that [10] did for Weyl geometry.

We therefore assume:

Postulate 2: Quantities of vanishing conformal weight comprise the class of physically meaningful observables.

For a field with nontrivial Weyl weight to have any physical meaning, it must be possible to construct weightless scalars by combining it with other fields. One situation in which this is easy to accomplish is the case of conjugate fields. We can use the symplectic form of biconformal space to generate fields in canonically conjugate pairs. Because the symplectic bracket, defined below, is dimensionless, such pairs are also conformally conjugate. This property holds for both the conformal and superconformal groups. With this property, the product of a field and its conformal/canonical conjugate always provides a measurable quantity and we are guaranteed to have measurable consequences even of weightful fields. We also note that zero-weight fields are self-conjugate.

3.1   The biconformal bracket

We make the preceding comments concrete as follows. First, we examine the symplectic structure of biconformal spaces.

The symplectic form, $\boldsymbol{\Theta} \equiv \omega^a \omega_a$, defines a symplectic bracket for the space. Let $\boldsymbol{\Theta}$ have components Θ_{MN}, and inverse Θ^{MN}, in a coordinate basis $u^M = \left(x^\alpha, y^\beta\right)$. Then we define the biconformal bracket of two fields f and g, to be

$$\{f, g\} \equiv \Theta^{MN} \frac{\partial f}{\partial u^M} \frac{\partial g}{\partial u^N}. \tag{6}$$

For a real solution to the field equations, the fields are conjugate if they satisfy the *fundamental biconformal bracket relations*,

$$\{f, g\} = 1 , \tag{7}$$
$$\{f, f\} = \{g, g\} = 0. \tag{8}$$

These relationships are analogous to the standard Poisson bracket relationships of classical mechanics.

For σ_C-representations, ω is a pure imaginary 1-form since it is defined to be the dual to the dilatation generator, D, which is pure imaginary (See Appendices A, B, and C). Consistent with this, we see that under complex conjugation,

$$
\begin{aligned}
\overline{\omega^a \omega_a} &= \bar{\omega}^a \bar{\omega}_a \\
&= \eta^{ab} \omega_b \eta_{ac} \omega^c \\
&= -\omega^a \omega_a
\end{aligned}
\tag{9}
$$

and we conclude that the dilatational curvature and the symplectic form are imaginary. That D is imaginary while P_a and K^a are complex conjugates of one another is further confirmed by the form of the supersymmetric structure equations ([17], [22]). It is of central importance to our results that the use of a complex gauge vector is consistent with real gauge transformations. This fact follows from the particular form of the conformal Lie algebra, and is not true of the Poincaré or Weyl algebras. Further discussion is presented in Appendix C.

As a result of Eq.(9), the fundamental brackets take the form

$$
\{f, g\} = i \, ,
\tag{10}
$$
$$
\{f, f\} = \{g, g\} = 0.
\tag{11}
$$

Note that since $d\omega$ is the defining symplectic form, these relationships are consistent with conformal weight. For the fields f and g given above, we have

$$
w_f = -w_g.
\tag{12}
$$

4   Motion in biconformal space

We now come to the description of motion in a biconformal geometry. The case of classical motion has been discussed in detail in [23]. The analysis of [23] establishes the phase space interpretation of a real, 6-dim biconformal geometry. Here we develop similar salient features for generic biconformal spaces and add further development. Following this, we turn to our discussion of quantum mechanics.

The classical/quantum distinction hinges on the choice of our final postulate and on representation. Postulate 3A leads to Hamiltonian dynamics (in any representation) while Postulate 3B leads to quantum mechanics in a σ_C-representation. Both formulations rely on identifying the integral of the Weyl vector with a multiple of the classical action, S.

In an arbitrary biconformal space, we set either

$$\frac{1}{\hbar}S = \frac{1}{\hbar}\int Ld\lambda = \int \omega = \int \left(W_\alpha dx^\alpha + \tilde{W}_\alpha dy^\alpha\right) \tag{13}$$

or

$$\frac{i}{\hbar}S = \frac{i}{\hbar}\int Ld\lambda = \int \omega = \int \left(W_\alpha dx^\alpha + \tilde{W}_\alpha dy^\alpha\right), \tag{14}$$

where we take the proportionality constant to be Planck's constant and the second form holds in a σ_C-representation for the conformal group. As we shall see, the value of this constant has no effect on the classical model, while it will give agreement with experiment in the quantum case. Notice too, we have written the Lagrangian with an arbitrary parameter λ since the integral of the Weyl 1-form is independent of parameterization. Finally, we observe that the gauge freedom inherent in the Weyl vector is consistent with known freedom in the action since adding a gradient to $W_A = (W_\alpha, \tilde{W}_\beta)$ is equivalent to adding a total derivative to the Lagrangian.

The integral of the Weyl vector, $\int \omega$, is the essential new feature of scale invariant geometries and as we show below, governs measurable size change. As shown in Appendix C, under parallel transport, the Minkowski length of a vector, V^a, changes by

$$l = l_0 \exp \int \omega \,, \tag{15}$$

where $l^2 = \eta_{\alpha\beta}V^\alpha V^b$. This change occurs because the Minkowski metric $\eta_{\alpha\beta} = diag\left(-1, 1, 1, 1\right)$ is not a natural structure of biconformal space. As a result, lengths computed using $\eta_{\alpha\beta}$ are not invariant in biconformal space. This is in contrast to lengths computed with the Killing metric K_{AB}. In general, as a result of the off-diagonal structure of the biconformal metric, K_{AB}, lengths computed with this biconformal metric are of zero conformal weight. In a σ_C-representation, the Weyl vector is imaginary, so the measureable part of the change in l is not a real dilatation – rather, it is a change of phase.

4.1 Postulate 3A: Classical mechanics

We achieve a formulation of Hamiltonian dynamics once we postulate:

Postulate 3A: The motion of a (classical) physical system is given by extrema of the integral of the Weyl vector.

Of course this is true of the action, but here we examine the variation in terms of the Weyl vector. Biconformal spaces are real symplectic manifolds, so the Weyl vector may be chosen so that the symplectic form satisfies the Darboux theorem for symplectic manifolds [24], [25],

$$\omega = W_\alpha \mathbf{d}x^\alpha = -y_\alpha \mathbf{d}x^\alpha \tag{16}$$

where x^α and y_α are coordinates on the space. For σ_C-representations, the Darboux theorem still holds, with

$$\omega = W_\alpha \mathbf{d}x^\alpha = -iy_\alpha \mathbf{d}x^\alpha \tag{17}$$

The classical motion is independent of which of these forms we choose.

Illustrating with the σ_C-case, the symplectic form is

$$\mathbf{\Theta} = \mathbf{d}\omega = -i\mathbf{d}y_\alpha \mathbf{d}x^\alpha \ . \tag{18}$$

As a result of this form, the pair (x^α, y_α) satisfies the fundamental biconformal bracket relationship

$$\{x^\alpha, y_\beta\} = i\delta^\alpha_\beta \ . \tag{19}$$

It is straightforward to show canonical transformations preserve this bracket whether the i is present on the right or not.

From Eq.(19) it follows that y_β is the conjugate variable to the position coordinate x^α and in mechanical units we may set $y_\alpha = \alpha p_\alpha$, where p_α is momentum and α is a constant with units of inverse action. Then,

$$i\alpha S \ = \ \int_c \omega \tag{20}$$

$$= \ -i\alpha \int_c \left(p_0 dt + p_i dx^i \right) . \tag{21}$$

It is important to note that for the classical results, α may any constant with appropriate dimensions. The classical theory does not contain any particular choice of natural constants.

4.1.1 One particle mechanics

In keeping with the usual assumptions of non-relativistic physics, we require t to be an invariant parameter so that $\delta t = 0$. Then varying the corresponding canonical bracket we find

$$
\begin{aligned}
0 &= \delta \{t, p_0\} \\
&= \{\delta t, p_0\} + \{t, \delta p_0\} \\
&= \frac{\partial (\delta p_0)}{\partial p_0}.
\end{aligned}
\tag{22}
$$

Thus, δp_0 depends only on the remaining coordinates, $\delta p_0 = -\delta H (y_i, x^j, t)$; the existence of a Hamiltonian is seen to be a consequence of choosing time as a non-varied parameter of the motion.

Applying the postulate, classical motion is given by $\delta S = 0$. Variation leads to,

$$
\begin{aligned}
0 &= i\alpha \delta S = -i\alpha \int \left(\delta p_0 dt + \delta p_i dx^i - dp_i \delta x^i \right) \tag{23} \\
&= -i\alpha \int \left(-\frac{\partial H}{\partial x^i} \delta x^i dt - \frac{\partial H}{\partial p_i} \delta p_i dt + \delta p_i dx^i - dp_i \delta x^i \right), \tag{24}
\end{aligned}
$$

which immediately gives us Hamilton's equations for the classical paths.

$$
0 = -\frac{\partial H}{\partial p_i} dt + dx^i, \tag{25}
$$

$$
0 = -\frac{\partial H}{\partial x^i} dt - dp_i . \tag{26}
$$

Notice that even if i and α are present initially, they drop out of the equations of motion.

4.1.2 Multiparticle mechanics

We revist Postulate 3A when more than one particle is present. In the case of N particles, the action becomes a functional of N distinct curves, $C_i, i = 1, \ldots, N$

$$
i\alpha S = \sum_{i=1}^{N} \int_{C_i} \omega. \tag{27}
$$

13

As for the single particle case, the invariance of time constrains p_0. However, since $\omega = -y_\alpha \mathbf{d}x^\alpha$ is to be evaluated on N different curves, there will be N distinct coordinates x_n^α and momenta, p_α^n. We have

$$
\begin{aligned}
0 &= \delta\left\{x_m^0, p_0^n\right\} \\
&= \left\{\delta x_m^0, p_0^n\right\} + \left\{x_m^0, \delta p_0^n\right\} \\
&= \frac{\partial\left(\delta p_0^n\right)}{\partial p_0^k} \frac{\partial x_m^0}{\partial x_k^0}
\end{aligned}
\tag{28}
$$

Now, since time is universal in non-relativistic physics, we may set $x_m^0 = t$ for all m. Therefore, $\frac{\partial x_m^0}{\partial x_k^0} = 1$ and we have

$$
\frac{\partial\left(\delta p_0^n\right)}{\partial p_0^k} = 0
\tag{29}
$$

which implies that each p_0^n is a function of spatial components only,

$$
p_0^n = p_0^n\left(x_k^i, p_i^k\right)
$$

This means that each p_0^n is sufficiently general to provide a generic Hamiltonian. Conversely, any single N-particle Hamiltonian may be written as a sum of N identical Hamiltonians,

$$
H = \frac{1}{N} \sum_{n=1}^{N} H
$$

so that eq.(27) becomes

$$
\begin{aligned}
i\alpha S &= \sum_{i=1}^{N} \int_{C_i} \omega \\
&= -i\alpha \sum_{i=1}^{N} \int_{C_i} \left(-p_0^n dt + p_i^n dx_n^i\right) \\
&= i\alpha \int_{C_i} \left(H\left(x_k^i, p_i^k\right) dt - \sum_{i=1}^{N} p_i^n dx_n^i\right)
\end{aligned}
$$

The introduction of multiple biconformal coordinates has consequences for the biconformal structure equations, for once we write the dilatational gauge field as

$$
\omega = -i\alpha \sum_{i=1}^{N} p_\alpha^n \mathbf{d}x_n^\alpha = -i \sum_{i=1}^{N} y_\alpha^n \mathbf{d}x_n^\alpha
$$

14

then

$$\mathbf{d}\omega = -i \sum_{i=1}^{N} \mathbf{d}y_\alpha^n \mathbf{d}x_n^\alpha$$

and the structure equation

$$\mathbf{d}\omega = \omega^a \omega_a$$

must be modified to include the proper number of degrees of freedom. We therefore write

$$\mathbf{d}\omega = \omega_n^a \omega_a^n$$

The remaining structure equations are satisfied by simply making the same replacement, $(\omega^a, \omega_a) \rightarrow (\omega_n^a, \omega_a^n)$. Thus we see that the introduction of multiple particles requires multiple copies of biconformal space, in precise correspondence to the introduction of a $6N$-dim (or $8N$-dim) phase space in multiparticle Hamiltonian dynamics. Here, however, the structure equations *require* this extension.

4.1.3 Measurement

In the presence of non-vanishing dilatational curvature, we can consider a classical experiment in which we could hope to measure relative size change with a real Weyl vector (or relative phase change with an imaginary Weyl vector). Suppose a system of length, l_0, moves dynamically from the point x_0 to the point x_1, along an allowed (i.e., classical) path C_1. In order to measure a relative size change, we *must* employ a standard of length, i.e., a ruler with length, λ_0. Suppose the ruler moves dynamically from x_0 to x_1 along any classical path C_2. If the integral of the dilatational curvature over surfaces bounded by C_1 and C_2 does not vanish, the relative sizes of the two objects will be different and form a direct contradiction with macroscopic observation. That is, since the ruler is our standard, any observed size change is attributed to the object. This difference is determined by the integral of the Weyl vector, as discussed above. Therefore, for our object and ruler, the new ratio of lengths will be given by the gauge independent quantity,

$$
\begin{aligned}
\frac{l}{\lambda} &= \frac{l_0}{\lambda_0} \exp \int_{C_1 - C_2} \omega = \frac{l_0}{\lambda_0} \exp \oint \omega \\
&= \frac{l_0}{\lambda_0} \exp \int \int_S \mathbf{d}\omega,
\end{aligned}
\tag{30}
$$

where S is any surface bounded by the closed curve $C_1 - C_2$. It is important to note at this point that though we are discussing two distinct systems (the ruler and the object) there is only *one* Weyl vector. That is, the Weyl vector is a one-form specified over the whole space, from which we see that

$$\int\int_S \mathbf{d}\omega = 0$$

by the existence of Hamilton's principal function. Therefore, no dilatations are observable along classical paths. This result also holds whether the Weyl vector is real or imaginary.

If the integral of the Weyl vector is a function of position, independent of path, then it is immediately obvious that no physical size change could be measurable! Consider our original comparison of a system with a ruler. If both of them move from x_0 to x_1 and the dilatation they experience depends only on x_1, then at that point they will have experienced identical scale (or phase) factors. Such a change is impossible to observe from relative comparison.

For the real case this observation can also be formulated in terms of the gauge freedom. Since we may write the integral of the Weyl vector as a function of position, the integral of the Weyl vector along every classical path may be removed by the gauge transformation [23]

$$e^{-\alpha\mathcal{S}(x)}. \tag{31}$$

Then in the new gauge,

$$
\begin{aligned}
\int W'_\alpha dx^\alpha &= \int \left(W_\alpha - \alpha\partial_\alpha\mathcal{S}(x) \right) dx^\alpha \\
&= \int W_\alpha dx^\alpha - \alpha\mathcal{S}(x) \\
&= 0
\end{aligned}
\tag{32}
$$

regardless of the (classical) path of integration. Note that we have removed all possible integrals with one gauge choice. It again follows that no classical objects ever display measurable length change. In the complex case, the phase changes cannot be removed by gauge choice, but they are nonetheless unobservable.

We now turn to an alternative postulate for motion, which leads to quantum mechanics.

4.2 Quantum mechanics

We have shown that there is no measurable size change along classical paths in a biconformal geometry. For systems evolving along other than extremal paths, however, dilatation may be measurable. Historically, physical dilatation has been the principal difficulty with the interpretation of Weyl and conformal geometries. In his original theory of electromagnetism, Weyl equated the gauge vector of dilatations to the electromagnetic vector potential. This equation is unsatisfactory because it implies, for example, substantial broadening of spectral lines in the presence of electromagnetic fields. Research over the following decade replaced the scale invariance with $U(1)$ invariance, paving the way for modern unitary gauge theories.

As we have shown, identification of the Weyl gauge vector as the Lagrangian leads to a satisfactory classical theory. Measureable dilatation then requires non-classical motion. Since such motion is claimed to occur in quantum systems, we may ask the following question: can quantum phenomena be understood as in some sense due to observable dilatations? Answering this question requires a formulation of measurement in a scale invariant geometry that can be compared to the rules for quantum measurement. First, we require a law of motion that allows non-classical paths in a generic Weyl or conformal geometry. For this, we postulate a probabilistic time evolution, weighted by dilatation, so that the classical paths remain the most probable. Second, we need a gauge invariant way to compare magnitudes in such a theory. To build a gauge invariant quantity we average over paths to get a probability for a system evolving from one location to another. Then we compare magnitudes according to postulate 2, forming a ratio with an appropriate standard.

With these ideas in place, we are in a position to compare the biconformal and quantum theories. We find that when the geometry is a σ_C-biconformal space, the description is indeed in agreement with quantum prediction – the imaginary Weyl vector produces measurable phase changes in exactly the same way as the wave function, and the use of a standard requires the probability to be expressed as the product of conjugate probability amplitudes. We are therefore justified in answering the question in the affirmative – in a σ_C-representation, measurement of dilatations may be identified with quantum effects. Thus, our model describes quantum measurement as the result of measurement of non-classical motion in a (scale-invariant, phase-space-like) σ_C-biconformal geometry.

Note that real and complex systems are solutions to the *same* set of biconformal structure and field equations. Since our goal is to formulate a theory of measurement for generic biconformal spaces, both sets of solutions must be interpreted according to the same rules, even though measurements of real and σ_C-reps look very different. In the first case, the connection is real so motion produces physical size changes. In the second, we have measurable phase changes. Because the real case is more directly geometric, we formulate the measurement theory for real solutions and then require that these same rules apply in the σ_C-case.

Postulate 3A for classical motion assumes that a system evolves along a unique path. If instead we take the more epistemologically sound view that if we do not measure the particle we do not know where it is, we can arrive at a more correct result. Thus, we assume size change is not impossible, but merely an improbable event – that in some sense, changing size requires some sort of dynamical effort. With this in mind we postulate [10]:

Postulate 3B: The probability that a system will follow any given infinitesimal displacement is inversely proportional to the dilatation the displacement produces in the system.

Because this form of the postulate is gauge dependent, it is useful to state the postulate in the following gauge-invariant way, found by integrating the previous form around a closed path:

Postulate 3B (invariant form): The relative probability of a system evolving along two paths, C_1, C_2 with common endpoints is

$$P = \min\left\{e^{\oint_{C_1-C_2}\omega}, e^{\oint_{C_2-C_1}\omega}\right\}.$$

We use this postulate to derive measurable correlates of motion. Note that biconformal space provides us *a priori* with the existence of a probability: the probability $P_{AB}(M)$ of finding a value M at point B for a system which is known to have had a value M_0 at point A. Finding $P_{AB}(M)$ is tantamount to finding the fraction of paths the system may follow which lead to any given value of M.

4.3   Motion with the postulate 3B

We would like to use the third postulate to predict the outcome of motion of a physical body. To begin our investigation, we will consider the motion

of a one dimensional object. Consider a rigid rod of initial length l_0, located at point x_0^i at time t_0. For simplicity we first consider only spacetime (x^α) displacements, neglecting any momentum (p_α) dependence of the problem. We wish to find the probability that the rod will arrive at a point x_1^i at time t_1, via a path C. However a number of issues present themselves at this point. First, we recognize that every measurement of a physical magnitude is a comparison. That is, at a given point, we cannot measure the size of an object without locally comparing it to a standard. For the present example, let our standard of length be a ruler of length λ_0. Then our physical prediction will involve only the dimensionless ratio,

$$\frac{l_0}{\lambda_0}. \tag{33}$$

We apply postulate 3B to this ratio. Since the probability of an infinitesimal displacement is postulated to be inversely proportional to the change in length, the probability of following a curve C is inversely proportional to the total change in length along the curve. To make this precise, we define the probability density of the rod following a particular path, C, to be equal to the following ratio,

$$G(C) = \begin{cases} \frac{\lambda}{l} \cdot \frac{l_0}{\lambda_0} & \text{if the rod decreases in relative length} \\ \frac{\lambda_0}{l_0} \cdot \frac{l}{\lambda} & \text{if the rod increases in relative length} \end{cases} \tag{34}$$

Without loss of generality we may discuss,

$$G\left(C\right) = \frac{\lambda}{l} \cdot \frac{l_0}{\lambda_0}. \tag{35}$$

The effect of dilatation on the length of the rod is given (in an arbitrary, fixed gauge) by

$$l = l_0 \exp \int_C \omega \ . \tag{36}$$

Next, we want to use this probability density to predict the location of the rod at a later time. This probability is the average over all curves C with endpoint x_1^i,

$$\begin{aligned}
P(x_1^i) &= \int \mathcal{D}\left[x_C\right] \left(\frac{\lambda}{l_0} \cdot \frac{l_0}{\lambda_0} \exp\left(-\int_C \omega\right) \right) \\
&= \int \mathcal{D}\left[x_C\right] \left(\frac{\lambda}{\lambda_0} \exp\left(-\int_C \omega\right) \right).
\end{aligned} \tag{37}$$

19

So far we have said nothing about the ruler. If we wish to comment on the length of the system, we also have to concern ourselves with what happens to our standard of length. By definition, λ is constant, but the actual ruler must also evolve according to the rules of the geometry. Since our act of measurement assumes only a measurement of the ratio $\frac{l}{\lambda}$ at the initial and final points, we cannot know what path the ruler has taken. To resolve this dilemma, we also average over all possible routes, C', for the ruler. This gives a second, independent path integral, so the result is gauge invariant and a probability,

$$P\left(x_1^i\right) = \int \mathcal{D}\left[x_{C'}\right] \int \mathcal{D}\left[x_C\right] \left(\exp\left(\int_{C'} \omega\right) \exp\left(-\int_C \omega\right)\right). \qquad (38)$$

Notice the product of the two line integrals is properly gauge invariant:

$$
\begin{aligned}
\exp\left(\int_{C'} \omega\right) \exp\left(-\int_C \omega\right) &= \exp\left(\int_{C'} \omega - \int_C \omega\right) \\
&= \exp\left(\oint_{C'-C} \omega\right). \qquad (39)
\end{aligned}
$$

Furthermore, since the sets of paths C and C' are independent, the path integrals separate:

$$
\begin{aligned}
P\left(x_1^i\right) &= \int \mathcal{D}\left[x_{C'}\right] \exp\left(\int_{C'} \omega\right) \int \mathcal{D}\left[x_C\right] \exp\left(-\int_C \omega\right) \\
&= \mathcal{P}(x_1^i)\mathcal{P}(-x_1^i) \\
&= \mathcal{P}(x_1^i)\overline{\mathcal{P}}(x_1^i), \qquad (40)
\end{aligned}
$$

where $\overline{\mathcal{P}}(x)$ is simultaneously the probability amplitude of the conformally conjugate system reaching x_1 along a forward oriented path, and the conformal conjugate of $\mathcal{P}(x_1^i)$. Notice that the probability depends only on a dilatation factor which is the same for any object (or standard) having units of length. For this reason, the answer is expressible as the product of an amplitude with its conformal conjugate, rather than the product of two totally unrelated amplitudes. It is interesting to see how the use of probability amplitudes follows from our need to employ a standard of length, and further, that the contribution of this standard ruler enters the problem symmetrically with the object under study.

This construction has been developed for a generic scale invariant geometry, and the result Eq.(40) is similar to that obtained in [10]. However, as

discussed in the introduction, there are important differences. Performing the construction in biconformal space, the Weyl vector automatically takes the required momentum-dependent form. Because of the symplectic structure, conformal conjugacy is automatic. But the most important difference between the current formulation and that of [10] arises because the conformal algebra admits nontrivial σ_C-representations.

We immediately recognize quantum mechanics when we look at the dynamical and measurement theories in a σ_C-representation. Since, in this case, the Weyl vector is pure imaginary (up to gauge), each factor in Eq.(40) is a standard Feynman path integral. The Schrödinger equation follows from the Feynman path integral in the usual way ([5],[27]). This is a marked difference from [10] and real biconformal representations, in which the Weyl vector is real and Eq.(40) is comprised of Wiener path integrals. In contrast, here the phase invariance of a wave function, $\psi' = e^{i\phi}\psi$ is created by the σ_C-conformal invariance, $M' = e^{\lambda w}M$. The i in the Weyl vector is the crucial i noted by London in 1927 [9]. He showed that the complex valued length dilatation factor $e^{\int \omega}$ is proportional to the complex valued Schrödinger wave function. Shortly thereafter, Weyl used this fact to develop the $U(1)$ gauge theory of electromagnetism. Here the i occurs despite the fact that we are working with real dilatations due to a combination of two factors. First, we have chosen to use a σ_C-invariant representation of the conformal algebra, which makes the configuration space non-integrable (see Section 4.4 below) and the Weyl vector imaginary. Second, we have imposed a non-invariant metric structure on the spacetime subspace of biconformal space. Because this metric is not a natural biconformal structure, its covariant derivative does not vanish and there is no necessary reason for it to be dilatationally invariant.

Clearly, both the dilatation function and the biconformal path generically depend on both spacetime and on momentum variables. Therefore, the path integrals in Eq.(40) may immediately be generalized to the usual double path integral of quantum mechanics (see [27]).

$$\mathcal{P}(x_1^i) = \int \mathcal{D}\left[x_C\right]\mathcal{D}\left[y_C\right]\exp\left(\int_C \omega\right). \tag{41}$$

We now examine further properties of σ_C-symmetry and non-integrability.

4.4 Absence of a submanifold

In the biconformal gauging of $O(4, 2)$, the 8-dim base manifold is spanned by the solder, ω^a, and co-solder, ω_a, forms satisfying the pair of Maurer-Cartan structure equations

$$d\omega^a = \omega^a_b \omega^b + \omega \omega^a , \tag{42}$$

$$d\omega_a = \omega^b_a \omega_b + \omega_a \omega . \tag{43}$$

Inspection of Eq.(42) shows that for the flat space, the solder form is in involution [16]. The Frobenius theorem [28], [29] then guarantees the existence of a 4-dimensional submanifold within the 8-dimensional biconformal space. This space is spanned by the co-solder form ω_a and is obtained by setting the solder form ω^a equal to zero. This division of the space breaks the σ_2-symmetry between the solder and co-solder forms.

In a σ_C-representation, the biconformal gauging of the conformal group again gives rise to an 8-dimensional space, but in this case, the solder form is not in involution due to the fact that the solder and co-solder forms are complex conjugates. As a result, Eq.(43) is the conjugate of Eq.(42). If we attempt to set the solder form to zero then by conjugacy, the co-solder form is also zero. Clearly, in the case of any σ_C-representation, we will not have involution of the solder form. The failure of the space to break into space-like and momentum-like submanifolds results in a fundamental coupling between momentum and position in the space. This structure is indicative of a quantum non-integrability, with similarity to the Heisenberg uncertainty principle.

Suppose our symplectic form is written in Darboux form. Then the symplectic bracket of x^α and y_β is

$$\{x^\alpha, y_\beta\} = \Omega^{AB} \frac{\partial x^\alpha}{\partial x^A} \frac{\partial y_\beta}{\partial x^B} = i\delta^\alpha_\beta . \tag{44}$$

where $x^A = (x^\alpha, y_\beta)$. By virtue of the above bracket relation, y_α has units that are inverse to those of x^α. We can create a momentum variable by dividing the y_β coordinate by a constant with units of action, as stated before. However, the value of the constant is now measureable. We get agreement with experiment by choosing the momentum coordinate to be, $p_\alpha \equiv \hbar y_\alpha$. This gives Planck's constant the interpretation equivalent to that of the speed of light in special relativity. The speed of light gives time the geometric units of length, while Planck's constant gives momentum the geometric units of inverse length.

Note the similarity to canonical quantization, with $i\hbar$ times the Poisson brackets going to commutators of operators. Here we achieve the factor of $i\hbar$ automatically from the σ_C-representation.

$$\{x^\alpha, p_\beta\} = i\hbar \delta^\alpha_\beta \ . \tag{45}$$

The natural symplectic bracket of biconformal space resembles the Dirac bracket of quantum mechanics, although at this point x^α, p_β remain functions, not operators. However, observe that we may define an operator, on functions of x^α, as

$$\hat{p}_\alpha \ = \ \{p_\alpha, \cdot\} \tag{46}$$

$$= \ -i\hbar \frac{\partial}{\partial x^\alpha} \tag{47}$$

Thus, the standard form of the energy and momentum operators is a consequence of the biconformal bracket. Other standard elements of quantum mechanics – notably the usual uncertainty relation, the introduction of operators, Hilbert space, etc. – follow from the Feynman path integral. Thus, our formulation is fully equivalent to quantum mechanics.

5 Conclusion

We have developed a new interpretation for quantum behavior within the context of biconformal gauge theory based on the following set of postulates:

(1) A σ_C-biconformal space provides the physical arena for quantum and classical physics.

(2) Quantities of vanishing conformal weight comprise the class of physically meaningful observables.

(3) The probability that a system will follow any given infinitesimal displacement is inversely proportional to the dilatation the displacement produces in the system.

From these assumptions it was argued that an 8-dim σ_C-biconformal manifold is the natural space of classical and quantum behavior. The symplectic structure of biconformal space is similar to classical phase space and also gives rise to Hamilton's equations, Hamilton's principal function, conjugate variables, and fundamental Poisson brackets when postulate 3 is replaced by a postulate of extremal motion. On this phase space, we have shown that

classical paths do not produce dilatational size change. As a result, no un-physical size changes occur in macroscopic observation. This fact overcomes a long-standing difficulty in utilizing conformal symmetry. In addition, as a central premise of this work we claim dilatational symmetry is the key to understanding the difference between classical and quantum motion. While classical paths experience no dilatation, quantum motion does.

Using a σ_C-representation for the conformal group, we obtain a Weyl vector that is pure imaginary. Despite this complex gauge vector, dilatational symmetry behaves like a real scaling on dimensionful fields. As is character-istic of Weyl geometries, relative magnitudes are found to change when their paths enclose a nonzero dilatational flux. However, they do not experience real size changes. Rather, using the Minkowski metric and an imaginary Weyl vector, we find that the measurable dilatations are phase changes.

The presence of dilatation symmetry and the resulting spacetime phase changes lead us to quantum phenomena. The properties of biconformal space determine the evolution of Minkowski lengths along arbitrary curves. Com-bining this with the classically probabilistic motion of postulate $3B$, together with the necessary use of a standard of length to comply with postulate 2, we conclude that the probability of a system at the point x_1^α arriving at the point x_2^α is given by,

$$
\begin{aligned}
P\left(x_1^i\right) &= \int \mathcal{D}\left[x_{C'}\right] \exp\left(\int_{C'} \omega\right) \int \mathcal{D}\left[x_C\right] \exp\left(-\int_C \omega\right) \\
&= \mathcal{P}(x_1^i)\mathcal{P}(-x_1^i) \\
&= \mathcal{P}(x_1^i)\overline{\mathcal{P}}(x_1^i),
\end{aligned}
\tag{48}
$$

where we are performing a path average over all paths connecting x_1^α and x_2^α. Eq. (48) reproduces the standard Feynman path integral of quantum mechanics which is known to lead to the Schrödinger equation. It is the requirement of a length standard that forces the product structure in Eq.(48). As in [10], it is significant that in the biconformal picture, the superposition principle will still hold because of the linearity of the Schrödinger equation. In addition, Bell's inequalities [30] will still be satisfied since in each case the physical probability is computed as the conjugate square of the time evolved field.

In addition, we find that the σ_C-representation implies a lack of invo-lution on the biconformal base manifold. From this, we see a fundamental entanglement between the conjugate variables x and p. Transport around

a spacetime path necessarily seeps into the momentum sector of the space. By examining the fundamental biconformal brackets for these variables we obtain a relationship similar to the Heisenberg Uncertainty principle of standard quantum mechanics.

It is important to note that these results express certain measurable consequences of stochastic motion in a classical 8-dim biconformal geometry. The biconformal curvature and connections are determined from field equations following by variation of an action linear in the curvatures. These facts give us the tools to ask meaningful questions about quantum gravity. Specifically, any scale-invariant quantity involving the connection and curvatures of biconformal space is, according to postulate 2, an observable quantity. Furthermore, it is possible this development will have some connection to the results of loop quantum gravity, since loop variables are invariant quantities of biconformal space. These ideas will be the subject of further study.

We began this investigation by considering a gauge theory of the conformal group. It is interesting to note that all of gauge theory (including conformal) had its origin in Weyl's investigation into dilatational symmetry. The original Weyl theory was absorbed into quantum mechanics with the original scale freedom becoming invariance under unitary gauge transformations [31]. Both the Weyl and Schrödinger theory describe the same evolution of a field in time given a factor of i and the Kaluza-Klein framework used by London [9]. We claim that dilatational symmetry remains a key to physical insight.

Appendix A. The conformal group

The conformal group generators include Lorentz transformations, $M^a_b = -M_{ba} = \eta_{ac} M^c_b$, translations, P_a, special conformal transformations, K^a, and dilatations, D, satisfying the commutation relations:

$$
\begin{aligned}
\left[M^a{}_b, M^c{}_d \right] &= -\left(\delta^c_b M^a{}_d + \eta_{df} \eta^{ac} M^f{}_b + \eta_{bd} \eta^{ae} M^c{}_e - \delta^a_d M^c{}_b \right), \\
\left[M^a{}_b, P_c \right] &= -\left(\eta_{cb} \eta^{ad} P_d - \delta^a_c P_b \right), \\
\left[M^a{}_b, K^d \right] &= -\left(\delta^d_b \delta^a_c - \eta^{ad} \eta_{bc} \right) K^c, \\
\left[P_a, K^b \right] &= 2 M^b{}_a - 2 \delta^b_a D, \\
\left[D, K^b \right] &= K^b, \\
\left[D, P_a \right] &= -P_a .
\end{aligned}
\tag{49}
$$

Appendix B. σ_C-Representations

The conformal Lie algebra has two independent involutive automorphisms [14]. The first,

$$
\sigma_1 : (M^a_b, P_a, K^a, D) \rightarrow (M^a_b, -P_a, -K^a, D)
$$

identifies the invariant subgroup used as the isotropy subgroup in the biconformal gauging. The second,

$$
\sigma_2 : (M^a_b, P_a, K^a, D) \rightarrow \left(M^a_b, -\eta_{ab} K^b, -\eta^{ab} P_b, -D \right)
$$

may be chosen to be complex conjugation to define σ_C-representations of the algebra. That is, if we assume the generators to be complex, σ_C-representations have P_a and K_a as complex conjugates, while M^a_b is real and D pure imaginary.

As an illustration of this property, notice that while both $so(3)$ and $su(2)$ have involutive automorphisms, the existence of a σ_C-representation singles out $su(2)$. Thus, while

$$
\begin{aligned}
\left[J_i, J_j \right] &= \varepsilon_{ijk} J_k , & (50) \\
\left[\tau_i, \tau_j \right] &= \varepsilon_{ijk} \tau_k , & (51)
\end{aligned}
$$

are both invariant under

$$
\begin{aligned}
\rho &: (J_1, J_2, J_3) \rightarrow (-J_1, J_2, -J_3) \\
\rho &: (\tau_1, \tau_2, \tau_3) \rightarrow (-\tau_1, \tau_2, -\tau_3) ,
\end{aligned}
$$

26

where $[J_j]_{ik} = \varepsilon_{ijk}$ and $\tau_i = -\frac{i}{2}\sigma_i$ (where σ_i are the usual Pauli matrices), it is only with the complex representation that $\rho = \rho_C$:

$$\overline{(\tau_1, \tau_2, \tau_3)} = (-\tau_1, \tau_2, -\tau_3) = \rho\left(\tau_1, \tau_2, \tau_3\right).$$

We provide two examples of conformal representations with this property. First, we consider the covering group, $SU(2,2)$, whose Lie algebra is isomorphic to that of $O(4,2)$. We note that due to the local isomorphism between $Spin(4,2)$ and $SU(2,2)$, this algebra can be represented in a spinorial basis. We will employ the 4×4 Dirac matrices, with the following conventions. The Lie algebra $su(2,2)$ may be written in terms of Dirac matrices, γ^a, satisfying

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab} = 2\ diag\left(-1, 1, 1, 1\right), \tag{52}$$

where $a, b = 0, 1, 2, 3$. We also define,

$$\sigma^{ab} = -\frac{1}{8}\left[\gamma^a, \gamma^b\right], \tag{53}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{54}$$

where the full Clifford algebra has the basis,

$$\Gamma \in \left\{1, i1, \gamma^a, i\gamma^a, \sigma^{ab}, i\sigma^{ab}, \gamma_5\gamma^a, i\gamma_5\gamma^a, \gamma_5, i\gamma_5\right\}. \tag{55}$$

The conformal Lie algebra may be obtained from this set by demanding invariance of a spinor metric [32], Q, given by,

$$Q = i\gamma^0. \tag{56}$$

If we require,

$$Q\Gamma + \Gamma^\dagger Q = 0 \tag{57}$$

the generators of the conformal Lie algebra are found to be [17], [22],

$$M^a_{\ b} = \eta_{bc}\sigma^{ac}, \tag{58}$$

$$P_a = \frac{1}{2}\eta_{ab}\left(1 + \gamma_5\right)\gamma^b, \tag{59}$$

$$K^a = \frac{1}{2}\left(1 - \gamma_5\right)\gamma^a, \tag{60}$$

$$D = -\frac{1}{2}\gamma_5. \tag{61}$$

Choosing any real representation for the Dirac matrices, γ_5 is necessarily imaginary and it follows that under complex conjugation,

$$\bar{M}^a{}_b = M^a{}_b, \tag{62}$$
$$\bar{P}_a = \eta_{ab}K^b, \tag{63}$$
$$\bar{D} = -D \tag{64}$$

so the action of σ_C is realized.

Alternatively, we may write a complex function space representation of the conformal algebra as follows:

$$M^a_b = -\frac{1}{2}\left(z^a\frac{\partial}{\partial z^b} + \bar{z}^a\frac{\partial}{\partial \bar{z}^b} - z_b\frac{\partial}{\partial z_a} - \bar{z}_b\frac{\partial}{\partial \bar{z}_a}\right), \tag{65}$$

$$D = z^a\frac{\partial}{\partial z^a} - \bar{z}^a\frac{\partial}{\partial \bar{z}^a}, \tag{66}$$

$$P_a = \frac{\partial}{\partial z^a} + \left(\bar{z}_a\bar{z}^b - \frac{1}{2}\bar{z}^2\delta_a^b\right)\frac{\partial}{\partial \bar{z}^b}, \tag{67}$$

$$K_a = \frac{\partial}{\partial \bar{z}^a} + \left(z_a z^b - \frac{1}{2}z^2\delta_a^b\right)\frac{\partial}{\partial z^b}. \tag{68}$$

It is straightforward to check that the Lie algebra relations are satisfied, while σ_C is again manifest. In both of these examples only the generators are complex – the group manifold remains real.

In either of these representations, the Maurer-Cartan equations inherit the same symmetry under σ_C. In particular, the gauge vector of dilatations (the Weyl vector) is imaginary as further discussed in the text. To clarify this, we show the dilatations generated by an imaginary generator, D, nonetheless give a real factor as expected.

First, consider $su(2,2)$. Using a basis for the Dirac matrices in which

$$D = -\frac{1}{2}\gamma_5 = -\frac{1}{2}\begin{pmatrix} -\sigma_y & \\ & \sigma_y \end{pmatrix}, \tag{69}$$

$$\sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \tag{70}$$

we define the definite conformal weight spinors χ^A, ψ^B by

$$D\chi = \frac{1}{2}\chi, \tag{71}$$

$$D\psi = -\frac{1}{2}\psi \tag{72}$$

28

and immediately see

$$e^{\lambda D}\chi = e^{\frac{1}{2}\lambda}\chi, \tag{73}$$

$$e^{\lambda D}\chi = e^{-\frac{1}{2}\lambda}\chi. \tag{74}$$

For the complex function space representation of the conformal group, the dilatation generator takes the form

$$D = z^a \frac{\partial}{\partial z^a} - \bar{z}^a \frac{\partial}{\partial \bar{z}^a}. \tag{75}$$

In one dimension, setting $z = re^{i\varphi}$, it easily follows that,

$$D = -i\frac{\partial}{\partial \varphi}. \tag{76}$$

Therefore, in this representation, D measures the phase of a complex number. Homogeneous functions of z and $\bar{z}$ are then eigenfunctions and D measures the degree of the homogeneity. Thus, if

$$f(z, \bar{z}) = z^a \bar{z}^b \tag{77}$$

then

$$e^{\lambda D} f(z, \bar{z}) = e^{(a-b)\lambda} f(z, \bar{z}) \tag{78}$$

so we indeed have dilatations, with the weight of the function encoded into the total phase.

Similarly in multiple complex dimensions, we have

$$D = z^a \frac{\partial}{\partial z^a} - \bar{z}^a \frac{\partial}{\partial \bar{z}^a}. \tag{79}$$

so we can build up eigenfunctions from powers of the norms,

$$f_{\alpha - \beta} = \left(\sqrt{z^2}\right)^{\alpha} \left(\sqrt{\bar{z}^2}\right)^{\beta}$$

Then

$$D f_{\alpha - \beta} = D \left(z^2\right)^{\alpha/2} \left(\bar{z}^2\right)^{\beta/2} = (\alpha - \beta) f_{\alpha - \beta}. \tag{80}$$

Notice that the Hermitian inner product, $z^a \bar{z}_a$, is of weight zero, $D\left(z^a \bar{z}_a\right) = 0$.

Appendix C. Gauge transformations

As mentioned in Appendix B, although we work with a complex valued connection, the gauge transformations remain real. In particular, although our Weyl vector is pure imaginary, the symmetry of the space remains dilatational (i.e., real scalings). By our construction, a local gauge transformation is given by

$$\Lambda = M^a{}_b \Lambda^b{}_a + D\Lambda^0. \tag{81}$$

Note that Λ is complex since $\Lambda^b{}_a$, Λ^0 are the real parameters used to exponentiate the generators M (real) and D (imaginary), respectively.

It follows that the gauge transformation of the Weyl vector is

$$\delta\omega = -d\Lambda^0, \tag{82}$$

where Λ^0 is a real number. Therefore, it is possible to define a scale-covariant derivative of a definite-weight scalar field,

$$Df = df + k\omega f, \tag{83}$$

where k is the conformal weight of f. To see this is, in fact, a gauge invariant expression, we perform a dilatational gauge transformation. The weightful function changes by a real scaling

$$f' = f\exp(k\Lambda^0) \tag{84}$$

while the Weyl vector changes by

$$\begin{aligned} \omega' &= \omega + \delta\omega \\ &= \omega - d\Lambda^0. \end{aligned} \tag{85}$$

We have,

$$\begin{aligned} D'f' &= d(f\exp(k\Lambda^0)) + k\left(\omega - d\Lambda^0\right)f \\ &= \exp(k\Lambda^0)Df \end{aligned}$$

So the equation is covariant and the Maurer-Cartan structure equations are invariant under real scalings. Of course, in generic gauges the Weyl vector is complex, but the invariance of the structure equations under gauge transformations guarantees consistency.

It is finally worth noting that regardless of whether the Weyl vector is complex or pure imaginary,

$$\exp\left(\oint \omega\right) \tag{86}$$

remains a pure phase since the Weyl vector is pure imaginary in at least one gauge and the above expression is gauge invariant. Note that in a σ_C-geometry the complex Weyl vector can never be fully removed by a (real) gauge transformation.

Appendix D. First order solution to the structure equations

The Cartan structure equations for flat σ_C-biconformal space are given by

$$\mathbf{d}\omega_b^a = \omega_b^c\omega_c^a + 2\omega_b\omega^a, \tag{87}$$

$$\mathbf{d}\omega^a = \omega^c\omega_c^a + \omega\omega^a, \tag{88}$$

$$\mathbf{d}\omega = 2\omega^a\omega_a, \tag{89}$$

and their conjugates, where ω^a corresponds to translation generators, ω is the Weyl vector, and ω_b^a is the spin connection, and we have utilized the structure constants of the Lie algebra (see Appendix A).

A first order perturbative solution is given by

$$\omega_b^a = \left(\delta_e^a\eta_{cb} - \delta_c^a\eta_{eb}\right)x^c\mathbf{d}x^e + \left(\delta_e^a\eta_{cb} - \delta_c^a\eta_{eb}\right)y^c\mathbf{d}y^e, \tag{90}$$

$$\omega^a = \{\mathbf{d}x^a + i\mathbf{d}y^a \tag{91}$$

$$+ \left(-\frac{1}{2}x^a x_e + \frac{i}{2}\left(\delta_e^a x_c y^c - x^a y_e\right) + \frac{1}{2}y^a y_e\right)\left(\mathbf{d}x^e - i\mathbf{d}y^e\right)\}, \tag{92}$$

$$\omega = i\left(y_a\mathbf{d}x^a - x_a\mathbf{d}y^a\right). \tag{93}$$

References

[1] J. Klauder, Phase space geometry in classical and quantum mechanics, quant-ph/0112010.

[2] P.A.M. Dirac, *The Principles of Quantum Mechanics, Fourth Edition*, (Oxford University Press, New York, 1958).

[3] C. Zachos, T. Curtright, Phase space quantization of field theory, Prog. Theor. Phys. Suppl. 135 (1999) 244-258.

[4] N.M.J. Woodhouse, *Geometrical Quantization*, (Clarendon, New York, 1992).

[5] R. P. Feynman, A. R. Hibbs, *Quantum Mechanics and Path integrals* (McGraw-Hill, New York, 1965).

[6] F. Berezin, Sov. Math. Izv. 38 (1974) 1116; Sov. Math. Izv. 39 (1975) 363.

[7] J. Isidro, Darboux theorem and quantization, hep-th/0112032.

[8] J. Isidro, Duality, Quantum mechanics and (almost) complex manifolds, hep-th/0304143.

[9] F. London, Z. Phys. 42, 375 (1927).

[10] J.T. Wheeler, Quantum measurement and geometry, Phys. Rev. D 41 (2) (1990) 431-441.

[11] E. Nelson, *Quantum Fluctuations* (Princeton University, Princeton, NJ, 1987).

[12] Wheeler, J. T., SU(3) x SU(2) x U(1) as the residual gauge group of the spacetime metric choice, *The Vancouver Meeting – Particles and Fields '91*, edited by D. Axen, D. Bryman and M. Comyn, World Scientific (1992) 854-856; J. T. Wheeler, SU(3) x SU(2) x U(1)SU(3): The residual symmetry of conformal gravity, *Mathematical Aspects of Classical Field Theory*, Contemporary Mathematics, Am. Math. Soc., Edited by M. J. Gotay, J. E. Marsden, & V. Moncrief (1992) 635-644.

[13] D. Hochberg, J. T. Wheeler, Scale invariant metric from conformal gauge theory, Bartol Research Institute Preprint, BA-89-24, (1989).

[14] A. Wehner, A new conformal gauge theory, Dissertation, Utah State University, 2001.

[15] J.T. Wheeler, New conformal gauging and the electromagnetic theory of Weyl, J. Math. Phys. 39 (1998) 299.

[16] A. Wehner, J.T. Wheeler, Conformal actions in any dimension, Nuc. Phys. B 577 (1999) 380-406.

[17] L.B. Anderson, J.T. Wheeler, Biconformal supergravity and the AdS/CFT conjecture. Nuc. Phys. B 686 (2004) 285-309. hep-th/030911

[18] E. A. Ivanov, J. Niederle, Gauge formulation of gravitation theories. I. The Poincaré, de Sitter, and conformal cases, Phys. Rev. D25, No 1 (1982) 976.

[19] E. A. Ivanov, J. Niederle, Gauge formulation of gravitation theories. II. The special conformal case, Phys. Rev. D25, No 4 (1982) 988.

[20] E. Bergshoeff, M. de Roo, B. de Wit, Conformal supergravity in ten dimensions, Nuc. Phys. B217 (1983) 489-530.

[21] A. Einstein, Sitz. Ber. Preuss. Akad. Wiss. 26, 478 (1918), including Weyl's reply.

[22] M. Kaku, P.K. Townsend, P. van Nieuwenhuizen, Properties of conformal supergravity, Phys. Rev. D 19 (10) (1979) 3166.

[23] J.T. Wheeler, Gauging Newton's Law, hep-th/0305017, submitted for publication.

[24] R. Abraham, J. Marsden, *Foundations of Mechanics* (Addison Wesley, New York, 1985).

[25] J. Jose, E. Saletan, *Classical Dynamics, A Contemporary Approach* (Cambridge University Press, Cambridge, 1998).

[26] L.D. Landau, E.M. Lifshitz, *Mechanics (Third Edition), Course of Theoretical Physics, Volume 1*, (Butterworth Heinmann, Oxford, 1976).

[27] E. Abers, *Quantum Mechanics*, (Pearson Education, New Jersey, 2004).

[28] H. Flanders, *Differential forms with applications to the physical sciences*, (Dover Publications, New York, 1989).

[29] R. Wald, *General Relativity*, (University of Chicago Press, Chicago, 1984).

[30] J.S. Bell, Physics 1, (1965) 195.

[31] L. O'Raifeartaigh, *The Dawning of Gauge Theory*, (Princeton University Press, Princeton, 1997).

[32] J.P. Crawford, Cifford algebra: notes on the spinor metric and Lorentz, Poincaré, and conformal groups, J. Math. Phys. 32 (3) (1991) 576.